REPORT RESUMES

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A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE, THE MADISON PROJECT.

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THE "DEMONSTRATION PROJECT" DESCRIBED IN THIS REPORT REPRESENTED A MAJOR PART OF THE ACTIVITY OF THE SYRACUSE UNIVERSITY-WEBSTER COLLEGE "MADISON PROJECT," BEGINNING IN SEPTEMBER, 1961, AND EXTENDING THROUGH THE SUMMER OF 1967. THE GOALS AS CONCEIVED BY THE MEMBERS OF THIS PROJECT WERE (1) TO CREATE EXPERIENCES IN MATHEMATICS WHICH WOULD BE BETTER THAN THOSE CHILDREN USUALLY ENCOUNTER, (2) TO CARRY ON THIS ACTIVITY AS MUCH IN THE VIEW OF THE EDUCATIONAL COMMUNITY AS POSSIBLE, AND (3) TO GAIN SUCH UNDERSTANDING OF CURRICULUM AND INSTRUCTION AS CAN BE GLEANED FROM THIS TYPE OF CREATIVE "CURRICULUM INNOVATION" ACTIVITY. IN A NARROWER SENSE, THE PROJECT WAS CONCERNED WITH DEVELOPING SEQUENCES OF CLASSROOM LESSONS COMBINING MATHEMATICS AND SCIENCE AT THE LEVEL OF GRADES K-9 AND WITH WORKING WITH VARIOUS TEACHERS AND SCHOOL SYSTEMS IN IMPLEMENTING THIS PROGRAM IN ACTUAL CLASSROOMS. THIS PROJECT WAS AIMED AT SEVERAL RELATED OUTCOMES--(1) THE CREATION OF NEW POINTS OF VIEW AND THE RECOGNITION OF NEW VALUES AND NEW PURPOSES, (2) THE IMPROVEMENT OF THE STATE OF THE ART, (3) EXPANDING THE REPERTOIRE OF THE PRACTITIONER, (4) IMPROVING ACTUAL PRACTICE IN THE SCHOOLS, (5) THE PRODUCTION OF CURRICULUM MATERIALS, AND (6) THE RECORDING OF INNOVATIONS VIA AUDIOTAPE, VIDEOTAPE, AND 16MM FILM. STUDENTS INVOLVED HAVE INCLUDED CULTURALLY-PRIVILEGED CHILDREN, "ORDINARY" CHILDREN, AND CULTURALLY-DEPRIVED CHILDREN. IN EVERY CASE, THE MATHEMATICIAN AND THE CHILD HAVE USUALLY EMPLOYED THE PROCESS-ORIENTED APPROACH AS OPPOSED TO A FACT-ORIENTED APPROACH IN THE DEVELOPMENT OF SIGNIFICANT MATHEMATICAL CONCEPTS. THE TASK FOR THE PROJECT HAS BEEN TO DEVISE SUITABLE "PROCESS" EXPERIENCES AND TO GIVE TEACHERS THE NECESSARY EDUCATIONAL BACKGROUND TO COPE WITH THE DEMANDS OF THIS STYLE OF TEACHING. FROM THIS COLLABORATION BETWEEN TEACHERS AND SCHOOLS, THERE HAS EMERGED A VARIETY OF DIFFERENT CURRICULUMS. A DISCUSSION OF FIVE OF THESE VARIOUS CURRICULUMS IS INCLUDED. (RP)

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FINAL REPORT

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OFFICE OF EDUCATION

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A MODERN MATHEMATICS PROGRAM AS IT PERTAINS TO THE INTERRELATIONSHIP OF MATHEMATICAL CONTENT, TEACHING METHODS AND CLASSROOM ATMOSPHERE (THE MADISON PROJECT)

October 1967

U. S. DEPARTMENT OF
HEALTH, EDUCATION, AND WELFARE
Office of Education
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October, 1967

The research reported herein was performed pursuant to a contract with the Office of Education, U. S. Department of Health, Education, and Welfare. Contractors undertaking such projects under Government sponsorship are encouraged to express freely their professional judgment in the conduct of the project. Points of view or opinions stated do not, therefore, necessarily represent official Office of Education position or policy.

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1. Introduction

Orientation to the Present Report

The "demonstration project" described in this report represented a major part, but not all, of the activity of the Syracuse University-Webster College "Madison Project," beginning in September, 1961, and extending through the summer of 1967. In fact, this work continues through the present and into the future, but under other financial auspices.

The goal of this project is to create experiences in mathematics for school children which will be in some sense "better" than those the children usually encounter, to carry on this activity as much in the public view of the educational community as possible, and to gain such understanding of curriculum and instruction as can be gleaned from this sort of creative "curriculum innovation" activity.[1]

The actual Madison Project program began earlier than the U.S.O.E.-sponsored portion, namely in November, 1956, at the Madison School in the culturally-blighted "15th Ward" of Syracuse, New York. The program has thus evolved, in time, over a period of eleven years. Subsequent to November, 1956, Project work moved into middle class neighborhoods in Syracuse, and into such culturally-privileged areas as Weston, Connecticut, Scarsdale, New York, and Lincoln, Massachusetts. It has moved from areas as small as a single school (the Madison School) or a small town (Weston, Connecticut and Lincoln, Massachusetts), into operations on as large a scale as simultaneous programs to reach 10,000 teachers of grades K-8 in the Chicago Public Schools, 20,000 teachers of grades K-6 in the New York City Public Schools, and similar programs in the Los Angeles City Schools, the schools of San Diego County, and of the City of Philadelphia.

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[1] This sentence already contains a word which should elicit disagreement: in what sense do we mean "better"? Hopefully, the remainder of this report will clarify the sense in which the word "better" is used here, by the Madison Project, in 1967. Used by other people, at other times, or in other contexts the word will surely have quite different meanings. Indeed, few more serious obstacles confront education than the unmistakable disagreements over values, goals, and judgements. We shall try to be as clear as possible on the values and goals involved in the various Madison Project curricula.

Thus the activity with which this report is concerned goes beyond any single discipline, as these are usually conceived, and the report necessarily has a strong historical flavor, a "diffusion of innovation" flavor, and a mathematical aspect, as well as a concern for curriculum and instruction.

Such an activity could, conceivably, have a variety of outcomes. It could be aimed toward a final written report; it could be aimed toward the identification of relevant variables, or the measurement of such variables, or establishing relationships among such variables; it could be aimed at establishing verbally- or symbolically-coded generalizations; it could be aimed at the creation of new points of view and the recognition of new values and new purposes; it could be aimed at the further development of practitioners' maxims;[2] it could be aimed at the improvement of the state of the art; or at expanding the repertoire of the practitioner (creating new alternatives to traditional procedures); or at improving actual practice in the schools; or at the production of curriculum materials; or at the recording of innovations via audiotape, videotape, and 16 mm. motion picture film.

The present project was aimed at the last seven outcomes listed above; that is to say, it was aimed at:

> the creation of new points of view and the recognition of new values and new purposes;
>
> the further development of practitioners' maxims;
>
> the improvement of the state of the art;
>
> expanding the repertoire of the practitioner;
>
> improving actual practice in the schools;
>
> the production of curriculum materials;
>
> the recording of innovations via audiotape, videotape, and 16 mm. film.

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[2] The phrase is borrowed from Polanyi (77), pp. 30-31 and elsewhere.

In the view of many,[3] including the present author, the armamentarium of educational research and development had, in recent years, been allowed to become constricted to an unrealistically narrow range of methods. Almost total reliance seems to have been shifted to the identification of variables, their measurement, considerations of their interrelationships, and the establishment of verbally or symbolically coded generalizations. Much more is possible, and, in the present view, is necessary.

It is important to recognize that there is an inevitable antagonism between various of these outcomes. Thus, if one wishes to improve actual practice in schools, one will listen to, and act upon, various confidential statements (e.g., the inadequacies of certain personnel, the existence of power struggles, etc.) which cannot ordinarily be included in written reports, and which (while having some general implications) will be viewed mainly as specific unique events.[4] If we seek only "scientific generalizations" we necessarily blind ourselves to some of the unique aspects of matters which are often of decisive importance for the success of efforts in practical implementation.[5]

Consequently, the present written report was not intended to be, and is not, either the goal of the entire effort or the only channel of communication to the profession. More than 117 16 mm. films exist, showing actual classroom lessons.[6] An even larger number of audio-tape recordings of lessons exist.[7] At least seven studies relating to the use of Madison Project materials exist, five of them performed independently of the Madison Project by outside personnel or agencies.[8] Various aspects of Project work, objectives, philosophy, lesson plans, and course outlines exist, many of them readily available in professional journals.[9]

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[3] Cf. Hawkins (45).

[4] Cf. Stotland and Kobler (100), Davis (29), Schlesinger (85), Schrag (87), Schrag (86), and Smith and Geoffrey (97).

[5] Cf. Storm (99).

[6] Cf. Appendix B.

[7] Cf. Appendix C.

[8] Cf. Appendix A.

[9] Cf. Appendix D.

Since the Project was concerned with producing changes in actual classrooms in actual schools, it is even more important that, at the time of this writing (Spring/Summer 1967) it is possible to visit classrooms and to observe Madison Project lessons being taught, and to observe teacher-education workshops in Chicago, San Diego, New York City, Los Angeles, and Philadelphia.[10]

Thus no one who wishes to know about the work of the Madison Project is limited to what can be learned from the present written report.[10a]

Statement of the Central Problem

The basic "problem" with which this effort has been concerned will be stated broadly by a single proposition, illustrated by some examples, focused by two corollary principles, and given further definition by a more extended discussion of what was actually done in various specific instances.

A. The "Process" Interpretation of Mathematics

Mathematics is in fact a process. It is not a collection of facts, definitions, algorithms, or explicit procedures, although each of these will find its place in any effort to carry out the process of actually "doing mathematics." This process is the important thing, and not its "result" or "the answer."

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[10] For assistance in arranging visits, write to The Madison Project, Webster College, Webster Groves, Missouri 63119. (Telephone: Area Code 314, WO 8-0500.)

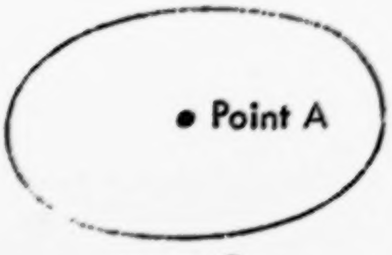
[10a] This point is emphasized in the belief that written reports usually do not, and possibly cannot, provide adequate means for the study of educational practices, whether innovative or traditional; furthermore, an over-emphasis on written reports can result in a preoccupation, at the operational level, with matters that will be reflected in such reports, and in a neglect of matters which will not be. Where the Project has neglected anything, interested persons may observe classes and workshops and identify such neglect for themselves.

There is nothing mystical about this, although our usual rhetoric copes with "outcomes" so much more easily than with "processes" that we frequently find ourselves thinking and acting as though the process is unimportant and the result is all-important, when the converse is more nearly true.[11]

If the "task" were arranging and implementing a pleasurable vacation, it would not be the actual selection of one beach instead of another, nor the return to work two weeks later, nor any such thing, that would constitute the all-important "goal"; it would be the process of "enjoying the vacation" that would be the important aspect of the matter.

Several specific examples mainly from current curriculum and instruction in contemporary mathematics, can indicate the distinction more clearly.

Example 1. "Inside" or "Outside." One "modern" textbook series includes questions of fact concerning whether the points A, B, and D are "inside" or "outside" of the corresponding curves C_A, C_B, C_D:



curve C_A

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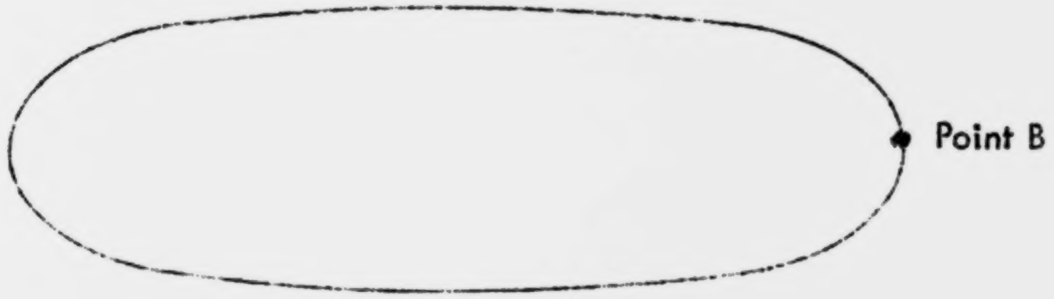
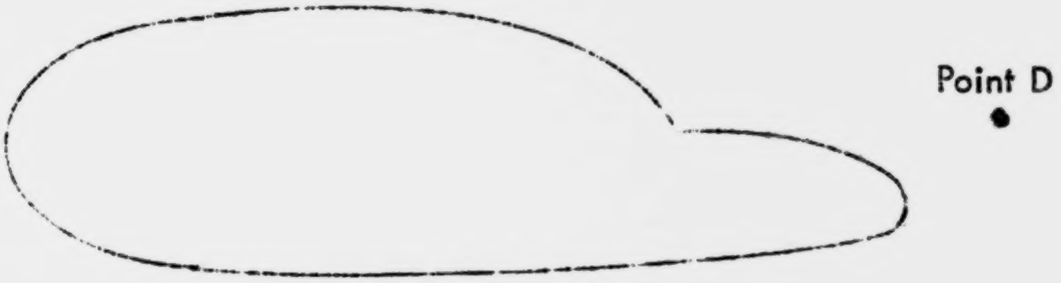
[11] Cf. Schwab (90), Hawkins (46), and Kurtz (65), "Help Stamp Out Non-Behavioral Objectives" (note that the word "Behavioral" in this title does not refer primarily to "behaviorism" as an epistemological doctrine, but rather to an orientation toward "process" rather than "product," an avoidance of a preoccupation with the memorization of facts and a greater emphasis on developing abilities "to behave"); also Polanyi (77), and Rosenthal (82). For an excellent example of a focus on "process," cf. Steiner (98).



curve C_B



curve C_D

Here one is clearly concerned with a matter of *fact*. The biggest single criticism that the Project, after eleven years of study, would make concerning schools is their almost exclusive preoccupation with verbally-coded statements of fact. Processes -- whether analyzing, persuading, criticizing, performing actual measurements, devising original laboratory experiments, conjecturing original mathematical theorems, or playing the guitar -- tend to be supplanted by some collection of verbal facts to be learned. This supplanting is what is here involved.

The theory is sometimes explicitly stated, either that students lack the ability to get beyond "facts," or else that, whatever your goals, "you must begin with facts."

It is the Madison Project's contention that neither of these statements is true. Quite the contrary: students *can* move beyond "facts" and deal with "processes," and many students perform better (and enjoy school more) if the school program focuses on "reasonable tasks" -- i.e., on *processes* -- and deals with facts incidentally as they relate to these processes.[12]

What process might be involved in the present example? Mrs. Doris Diamant Machtinger has worked out a sequence dealing with the "inside-outside" distinction that replaces an uninteresting, unchallenging, unexciting fact with a rewarding process of analysis and explication. She has found this sequence to be useful with capable primary grade children (K-3). It is unquestionably useful with intermediate grade children (4-6).

In Mrs. Machtinger's procedure, the children have the task of working together as a group, of stating criteria by which they would decide whether a point lies "inside" or "outside," and of criticizing the adequacy or inadequacy of the various specific criteria that are suggested.

The teacher has the extremely important task of interacting with the students skillfully during the discussion: the teacher will propose new instances to which the various criteria can be applied, will raise questions about ambiguous terms, and generally play a kind of "devil's advocate" role. For example, the teacher will present this example:

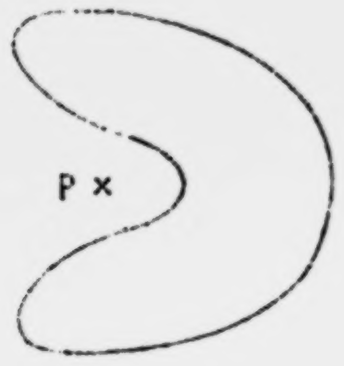


Figure 1. Is the point P "inside" of the curve?

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[12] It seems reasonable to conjecture that the replacement of "processes" by "facts" does not originate in any attribute of students, but rather from four attributes of traditional schools: i) traditional school procedures can teach facts more easily (and more cheaply) than processes; ii) traditional school procedures can test for facts more easily (and more cheaply) than they can (continued on next page)

The "answer" here is completely unimportant. In ordinary lay use of the words, either answer is defensible, depending upon whether we are thinking of questions such as "Is the fly inside the frog's mouth?" or else of questions such as "Is the horse inside the corral?" Thus, to the question "Is the point P inside the curve?" one is fully entitled to answer either "yes" or "no" -- provided one can reasonably defend whichever answer one gives.

This freedom within lay usage has, in fact, a parallel within mathematical usage, for in mathematics, also, there remains a question of interpretation. We can, for example (thinking of the frog's mouth) answer "yes, the point P is inside," and state such criteria as: "because any line through the point P will intersect the curve." If we develop this theory, we shall gradually construct an elementary portion of the theory of "convex hulls," which constitutes a respectable portion of mathematics.

Alternatively, we can answer "No, the point P is not inside the curve." Thinking of the horse and the fenced-in corral, we can say: "the horse is free to run away." This can lead to the mathematical idea of a "remote point" R,

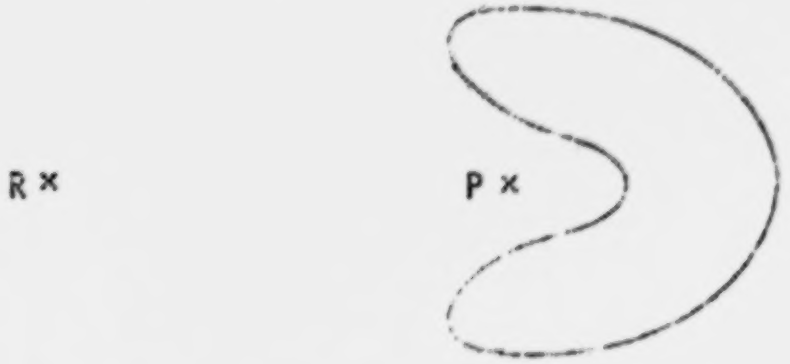

Figure 2. The point R is a "remote point."

(12 continued)
test for processes; iii) "facts" are less controversial and less threatening than "processes" (witness instances of teachers and students writing original plays, etc.); iv) the rhetoric and epistemological orientation of schools (possibly allied with the linear limitations of language which Suzanne Langer, Michael Polanyi, and Marshall McLuhan have remarked) can cope with facts more easily than with processes.

and to the criterion that "there is a path leading from the point P to the point R that does not cross the fence."

The teacher's role here would require her to ask some questions to cause the students to analyze further what they mean by a "remote point," and also to analyze further (and state reasonably clearly in words) what they mean by "path." There is no problem in agreeing that "fence" means the original curve. In this way, the students are engaging in the process of explicating[13] the ideas of "inside" and "outside" and developing some elementary portions of point-set topology, which is also a respectable branch of mathematics.

To the question "Is the point P inside the curve?" (in Figure 1) the children may correctly answer either "yes" or "no." The actual answer is unimportant, and the process of explication can be carried out in such a way as to defend either answer. What is important is the child's ability to carry out this process of explication.

The mathematically-knowledgeable reader can easily see how this process of explication can be carried further. The teacher can present an example such as

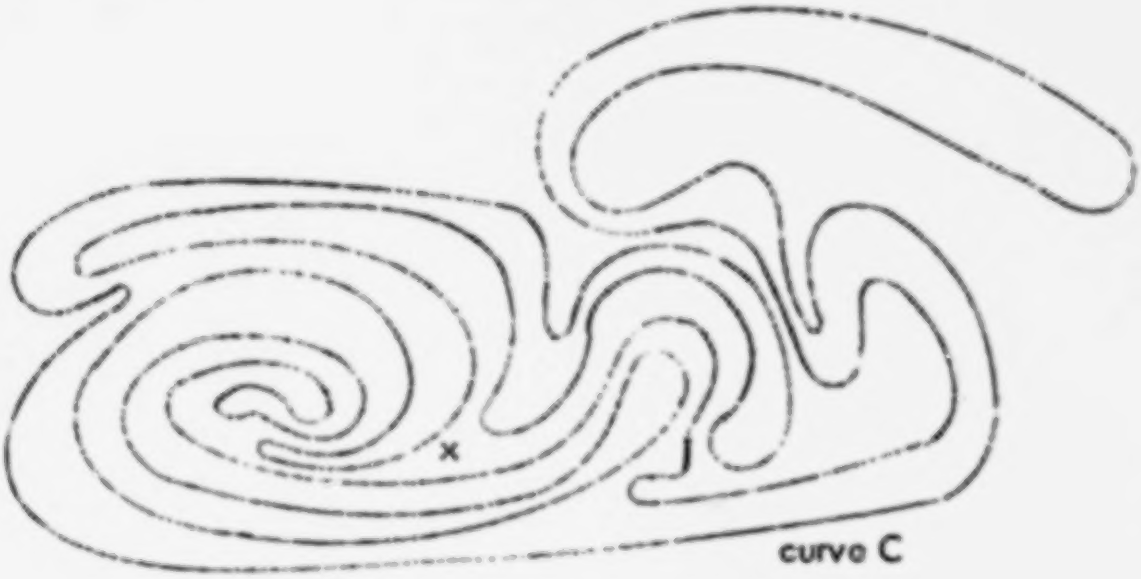

Figure 3. Is the point marked "x" inside the curve C?

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[13] Cf. the dictionary definition of "explicate" as "to unfold the meaning or sense of."

or such as

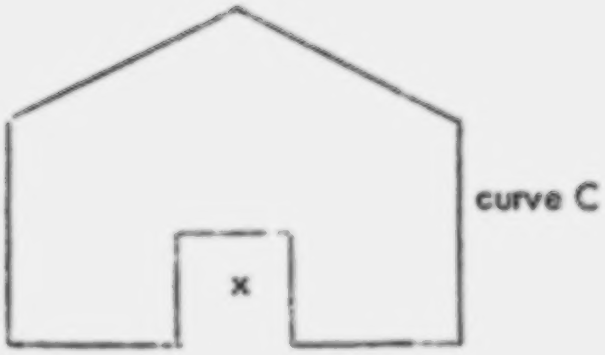

Figure 4. Is the point marked "x" inside the curve C?

or such as

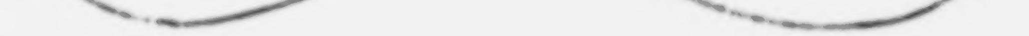

Figure 5. Is the point P "inside"?

or such as

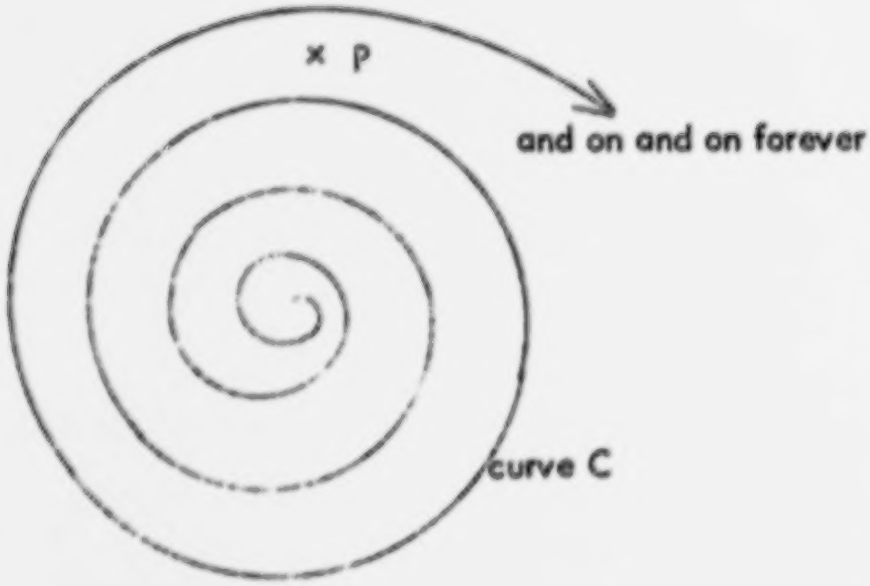
× P
and on and on forever
curve C

Figure 6. Is the point P inside curve C?

or such as

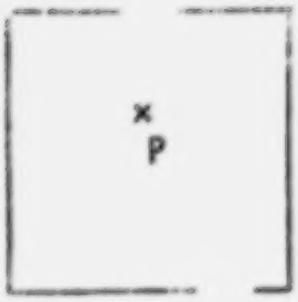

Figure 7. Is the point P "inside"?

By such examples, the teacher can exercise the students in the *process* of *explication*, including a recognition of the role of dimensionality, and the possible creation of a category labeled "does not apply" or "undecidable."

When one describes this process as "having children develop mathematics for themselves" one is met by the query: "Why not merely *tell* the students?" This, of course, replaces *process* by *fact*. We can *tell* the student how to play the piano, and *at a verbal level* he may now "know how to play the piano" *better* than if he actually practiced until he was able to play. What is at stake is not the most efficient route to an agreed-upon goal, but *finding* routes to *quite different goals*.[14] Possibly part of the *art* of explication is, as Polanyi argues, necessarily ineffable and able to be learned only by "do-it-yourself" procedures and by a kind of apprenticeship to one who has mastered the art.[15]

Remark 1. How can one exhibit a *process* for observation by those who wish to understand it? The Madison Project answer has been to reject anything in the nature of a "still photograph," which might *suggest* movement but could not actually *show* it, to reject "before" and "after" descriptions which might establish the results of a process but would necessarily *omit* the process itself, and instead to record actual lessons on audio-tape, video-tape, and 16 mm. film. This makes it possible for *anyone* to observe the *actual process of explication*, the role of the students in suggesting criteria, the role of the students in criticizing criteria that have been suggested, and the role of the teacher in clarifying, challenging, and being "devil's advocate" in a kind of "adversary dialogue."[16]

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[14] Admittedly, this does not answer Gagné's question as to whether it might not be more efficient to be more *explicit*. This is a very serious question indeed, and is surely not entirely resolved at present. Cf. Polanyi (77), and also Hawkins (45). G.A.Miller, Roger Brown, Aldous Huxley, and others have also discussed this.

[15] Peter Schrag, in the volume *Voices in the Classroom* (36), records excellent observations of schools and teachers in various parts of the United States and in various kinds of communities. It would probably be a defensible interpretation of Schrag's observations to argue that Schrag sees *real education occurring in precisely those schools where a process approach is somehow utilized*.

Remark 2. Polanyi argues that the explication of the message is possible only when we attend to the meaning carried by the message, and treat the language itself as "transparent." We must avoid becoming focally aware of the language itself; to do so would be to lose the meaning. Anyone knows this who has proofread printed material, and has become lost in the search for broken pieces of type with the consequence of failing to keep track of the content of the message. As Polanyi says, one may "read" a book, or one may "observe" a book, but one may not do both simultaneously.

Similarly, when viewing TV, persons familiar with television techniques may become lost in a focal concentration on the technology: How many cameras are they using? What lens complements does each camera have? Where are the cameras located? What "wipes" or "special effects" are being employed? Does the result show evidence of instantaneous editing? With or without a "special effects box"? Does the result show evidence of subsequent assembly by electronic editing?

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[16] This description of the teacher's *cognitive* role must not be misread as a description of the teacher's *affective* role. *The Project places great stress on the teacher wishing for the children to emerge the victors in a fair fight*. In one or two cases where teachers really sought to gratify their own personal needs by scoring *actual* victories over the children -- by "showing them who's boss around here" -- this method of teaching has failed catastrophically.

Moreover, in cases where the teacher was overly protective of the children and sought to make *them* the victors at the cost of a patently "rigged" contest, the method has also proved ineffective. This teaching strategy appears to require that the teacher treat the children as *respected opponents*, and that the teacher so arrange the tasks that the children will succeed in an "honest fight." This is possible, because the explication of meaning and the "discovery" or "invention" of viable mathematical structures *is, in fact, a possible task*.

The result of such a focal concentration on the "language" of the message is necessarily to lose track of the content of the message. One could not recount, say, the plot of the play.

A third example of focal concentration on the language of a message obscuring a comprehension of the content of the message occurs in stage-fright, which is, indeed, caused by precisely this focussing on the wrong aspect of the communication. Unless the language itself is transparent, the communication will be opaque as to content.

Metaphorically, one can compare this with the phenomenon of looking at or through the windshield of an automobile. If one focuses one's eyes so as to see the windshield, one cannot then use the windshield to look through. If one wishes to use the windshield in order to look through it at things outside, then one must avoid looking at the windshield itself. (The act being described here is more easily performed if the windshield is neither too clean nor too dirty.)

If Polanyi is correct, then the usual present school practice is very seriously in error. Schools ordinarily focus attention on the language itself, and thereby preclude the use of this language for the communication of significant message content. To use a language for significant communication one should use the language transparently, and not allow it to become a matter of focal attention.

This is precisely contrary to the usual practice in most school situations.

The matter is of special interest to Madison Project teachers, since outside observers have frequently commented upon how infrequently the Project defines or explains a word. This represents an intuitive judgement by Project teachers that it is ordinarily better to use words, and to allow children to explicate meanings for themselves, from contextual clues, etc.

Polanyi, if he is correct, indicates that while one can use some words to discuss others, only the words being used, and not those being discussed, can carry significant messages.

It is almost certainly true that children learn language more effectively outside of school than they do inside school. One has had for years the child saying "ain't" who was judged as somehow "less intelligent" than the child who said "isn't." Yet surely the same *intelligence* is required to master the use of either word. Is Polanyi's suggestion correct: "ain't" was learned because it was *used transparently*, whereas "isn't" was not learned because it was the subject of focal awareness? (Admittedly there are other important factors, such as values, status, social acceptance or rejection, ego-ideals, peer-group influence, and even simple frequency of occurrence.)

If Polanyi is correct, one cannot be simultaneously a "proofreader" and a "reader." Language is learned through use, and not by focal awareness. (By contrast, if you already know how to *use* the language, a focal awareness of language can allow you to learn *about* the language, as in the study of grammar and linguistics.)

The relevance or correctness of this remark is not presently known to us, and little or nothing in the present report, or in Madison Project practice, is dependent upon it.

Remark 3. It seems virtually inconceivable that any commercially-available elementary school textbook series in the next few years will emphasize *process* rather than *facts*. This is one of the reasons why Project teachers in general wish to see the textbook eliminated from its present role as tyrant of the classroom, and also why the Project has *not* chosen to intervene in the educational process at the level of writing textbooks.

Example 2. The Parallel Postulate. (This example is taken from the history of mathematics itself, and not from mathematics education.) Given a line L and a point P which does not lie on line L, how many lines are there through the point P that do not intersect line L?

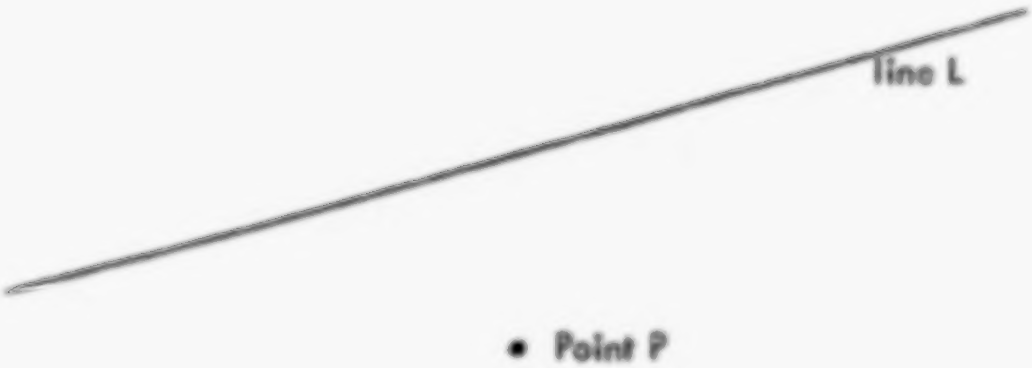

Figure 8. How many lines are there through Point P that do not intersect line L?

As is well known, in the usual lay (and school) sense of "right" and "wrong," there is no answer to this question which is uniquely "right," and three different answers none of which is actually "wrong."

One can answer: None. If this approach is properly developed, one gets a viable geometry for which the surface of a sphere can be used to provide a model.

One can answer: Exactly one. If this approach is properly developed, one gets the usual Euclidean plane geometry.

One can answer: Infinitely many. If this approach is properly developed, one gets a viable geometry for which the pseudosphere surface provides a model.

In any case, the "mathematics" does not lie so much in the "fact" or "answer" one, none, or infinitely many, as it does in the process by which one develops the relevant mathematical structure. It would probably not be appropriate to say that one develops the structure in order to "find" the "answer."

Example 3. Is the Plane Divided into Fourths? (This example is taken from a lesson observed in a suburban junior high school in the spring of

1967. The children were bright eighth-grade college-preparatory students who had been selected to spend their eighth-grade year studying ninth-grade algebra out of the popular Houghton-Mifflin book by Dolciani, et. al. This was not a Madison Project class.)

The question arose as to whether non-perpendicular lines divide the plane into fourths:

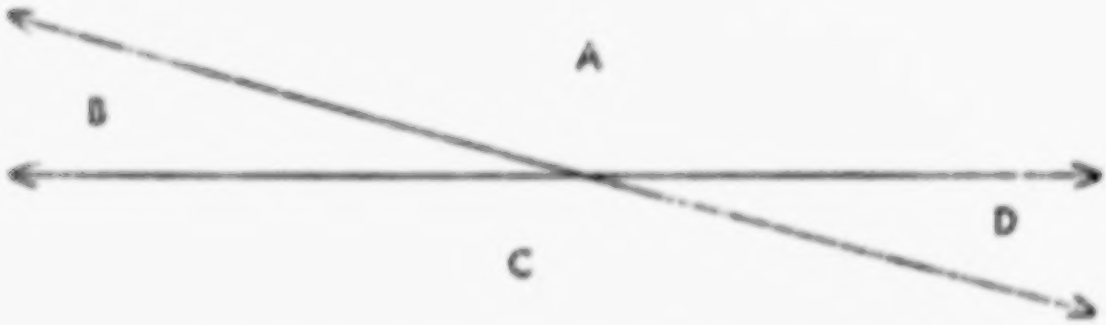

Figure 9. Is region A larger than region B?

We can put the question in the form (which the children also came to use during the course of the discussion): Is region A larger than region B? (Cf. Figure 9.)

Again, either answer -- yes or no -- can be effectively defended. What was impressive about this class discussion was that the children were process-oriented and wanted to work through the process of analysis and explication -- indeed, this was the first point in a previously fact-oriented lesson where the children showed any animation or interest. Both answers acquired able and articulate defenders, and an excellent discussion almost took place.

What prevented the discussion from occurring was that the teacher -- whose intentions were clearly the best -- was not adequately able to handle a process-oriented explication.

In support of the answer "Yes, region A is larger," two theories began to take shape. One argued that region B was congruent to a proper subset of region A. The teacher could have challenged this formulation by pointing out that region B is congruent to a proper subset of itself:

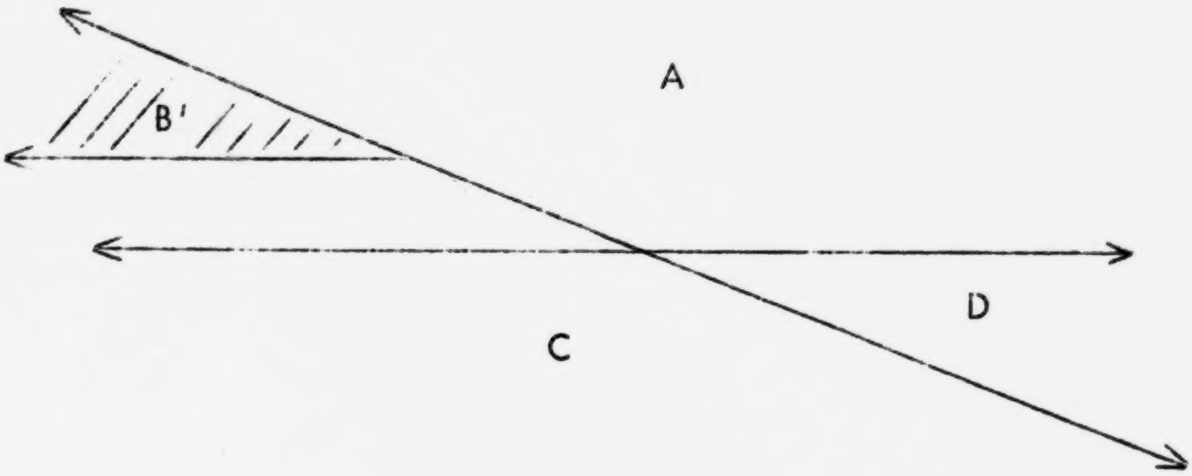

Figure 10. Region B (cf. figure 9) is congruent to region B′, which is a proper subset of region B.

The teacher seemed to lack facility in constructing this kind of "test-case" type of example, against which to test the adequacy of the various theories.

The second theory in support of the answer "Yes, region A is larger," again provided by a group of eighth-grade children, depended upon drawing a circle centered at the point of intersection (as in Figure 11),

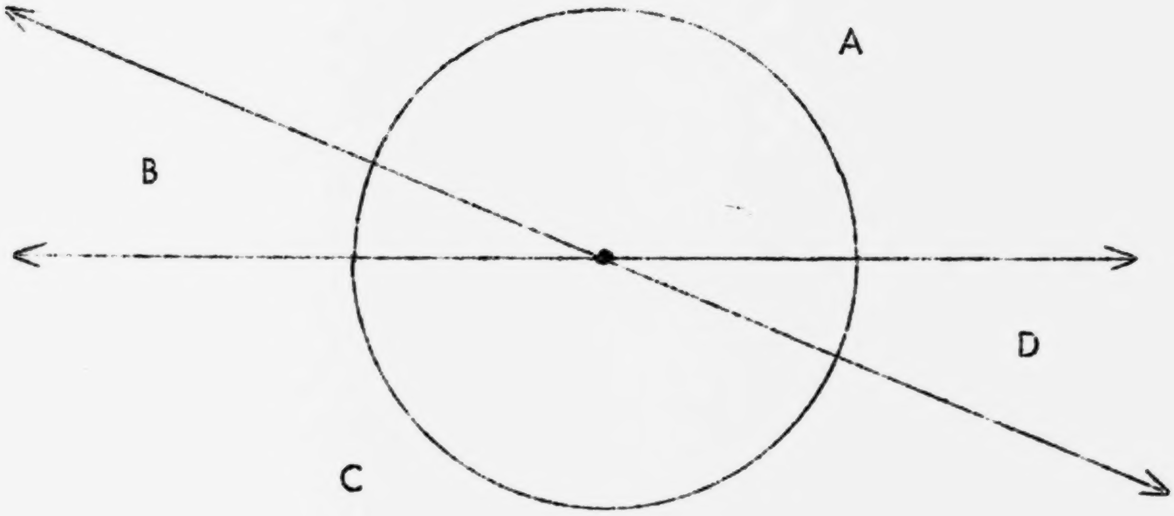

Figure 11.

and from this going on to imagine a sequence of circles, all concentric, with increasing radii. This theory could have been explicated to lead to the theory of infinite sequences, and to the distinction between *subtractive comparisons* vs. *ratio comparisons*.

In support of the answer "No, region A is not larger than region B" a theory of infinite sets began to be explicated.

This process is mathematics. *To carry out this process is to "do mathematics."* Admitting that these examples are chosen with malice aforethought, we can still note that for each question *either answer is defensible*. (Just, one might say, as one can decide to drive automobiles on the left side of roads, as the British do, or on the right-hand side, as we do. Either answer is "correct." The task is not to choose an answer, but rather *to work out* a consistent system.)

The *process of mathematics* is largely foreign to schools. It is replaced by the study of facts. Nearly everything of interest and value is thereby lost. The loss is unnecessary, since *children* are ready for -- and, in fact, *prefer* -- a "process" approach.

Example 4. Is There a Number Whose Square is 2? (This question arose in a 9th grade Madison Project class. The explication occurred during a sequence of lessons. As many of these as possible were recorded on video-tape, and kinescopes have been made. The resulting films bear the titles *Quadratic Equations*, *Introduction to Infinite Sequences*, *What is Convergence?*, and *Bounded Monotonic Sequences*, and are numbered, respectively, 110, 111, 112, and 113 in the list in Appendix B. The class was an all-girl class of college-capable ninth graders.

In support of the answer "no, there is none," a group of girls, admittedly with some help from the teacher, considered the number of factors of 2 in p, in q, in p^2, in q^2, and in $2q^2$, based on the assumption that

$$\left(\frac{p}{q}\right)^2 = 2 \ ,$$

thereby arriving at a contradiction (through a sequence of easy lemmas).[17]

Quite unexpectedly (to the teacher), another girl developed the following argument:

Draw a graph of the parabola $y = x^2$.

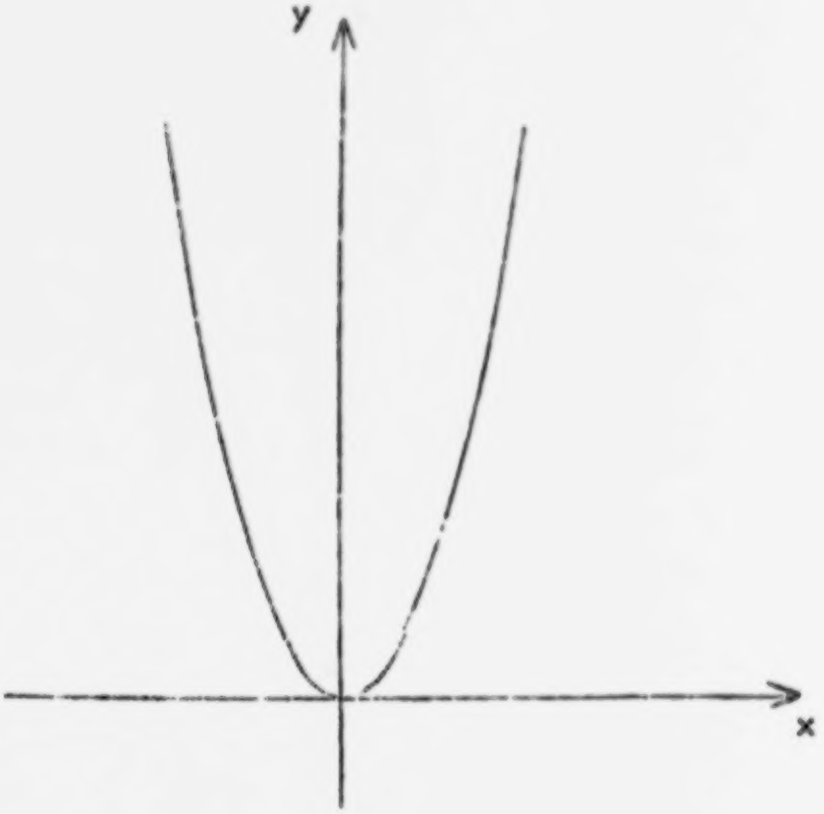

Figure 12. The parabola $y = x^2$.

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[17] For this same question, a 6th grade [sic!] boy, without help from the teacher, suggested considering the last non-zero digit in the decimal answer, and pointed out that if that was "1" the square would have a "1" (where it should have a zero), if the last digit was 2 the square would have a 4, and so on. Last digits of 3, 4, 5, 6, 7, 8, and 9 would yield, respectively, last digits in the square of 9, 6, 5, 6, 9, 4, and 1 -- but never the required zero!

Now draw the line $y = 2$.

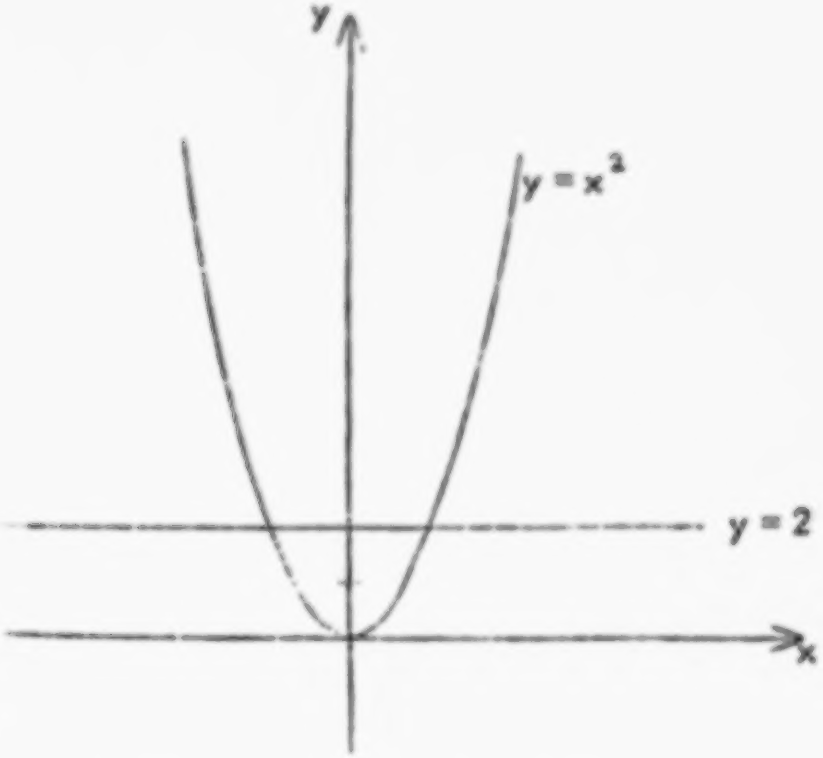

Figure 13. Do the graphs intersect?

Now the question is transformed into the question: do these two graphs intersect?

For the further explication of this problem, leading ultimately to the theory of bounded monotonic sequences and the real numbers, see Davis (22), and see also the films listed above.

Example 5. Father, Daughter, and the "New Mathematics." A father who is not a mathematician (but is a reporter and writer, and therefore especially able to see and report events) has a daughter studying "new mathematics" in a suburban school. The school program does not make explicit use of any Madison Project materials or techniques.

His daughter recently came to him saying that she couldn't remember the rule for dividing fractions.[18] What is interesting is that father, also, could not remember the rule. Were father and daughter therefore in the same situation? By no means; for some reason they were not, and this difference -- which is not explained in the present report because we cannot explain it with confidence -- is precisely the heart of much of the Madison Project's work.

The father began trying problems in division, to see if he could figure out "what you are doing when you divide." After looking at examples such as $8 \div 4$, $10 \div 2$, $18 \div 3$, and so on, he decided that you are answering the question "How many fours are there in eight?" and he represented this on a number line[19] as follows:

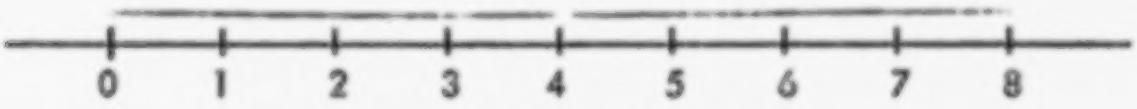

Figure 14. How many 4's in 8?

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[18] There is nothing basically new in this example. Warwick Sawyer argued ten years ago that "The poor teacher says 'Can't you remember the rule?' The good teacher says 'Can't you see the pattern?'"

[19] The nearly ubiquitous role of the "number line" in modern mathematics teaching has theoretical support from laboratory studies in psychology. George A. Miller (74) reports an experimental study by Coonan and Klemmer on what would ordinarily be called "absolute judgement," an important kind of discrimination ability that can be measured in the Wiener-Shannon unit known as the "bit." Klemmer and Coonan report a channel capacity of 3.9 bits for ability to locate points on a number line (in contrast, say, with "absolute judgement" of the pitch of a musical sound, the "saltiness" of a drink, the curvature of an arc, the loudness of a noise, etc.). This is the most satisfactory discrimination performance ever recorded for unskilled subjects (e.g., exempting musicians with perfect pitch, etc.). That is
(continued on next page)

He was then able to extend this process to problems such as

$$\frac{1}{2} \div \frac{1}{4}$$

and

$$5 \div \frac{1}{2}$$

and

$$4 \div \frac{1}{3} ,$$

etc.

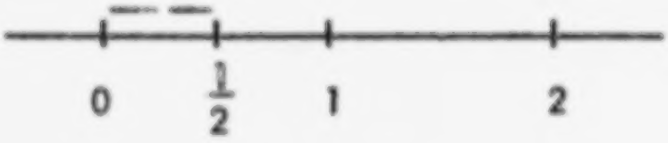

Figure 15. How many $\frac{1}{4}$'s in $\frac{1}{2}$?

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[19] (continued)
to say, people in general are very good at locating points on a number line. Of course, given the ubiquity of rulers, thermometers, ammeters, gasoline gauges, etc., one can suspect that superior performance on linear scaling is a result of massive practice. (Cf. Miller(74), pp. 85 and 86). Rosenthal (82) has some contrary evidence, suggesting that actually people are not that good at locating points on the number line.

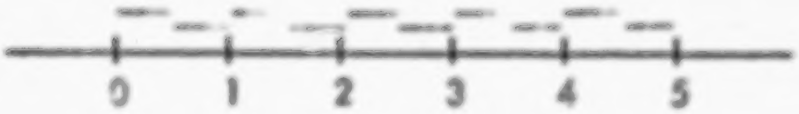

Figure 16. How many $\frac{1}{2}$'s in 5?

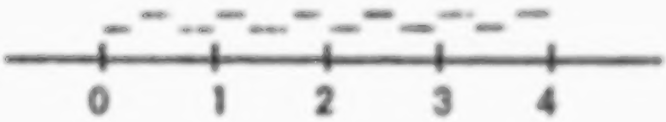

Figure 17. How many $\frac{1}{3}$'s in 4?

These pictures gave the answers:

$$\frac{1}{2} \div \frac{1}{4} = 2$$

$$5 \div \frac{1}{2} = 10$$

$$4 \div \frac{1}{3} = 12\ ,$$

from which father deduced[20] the familiar "invert and multiply" rule.

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[20] Or, more properly, "generalized to" the usual rule. Because of its unsatisfactory logical status, the process of "generalizing from instances" has nearly been banished from many mathematics classes -- yet it is surely one of the basic tools of creative mathematics.

What was there here that father was able to do that daughter could not -- and should not -- have done herself? This is one of the central questions in all "inductive," "discovery," "exploratory," or "explicational" learning. Is it something in the area of self-confidence (or self-concept) -- a belief in your own ability to work out the rule for yourself? Does school promote immature dependency upon authority? Is it a question of your concept of mathematics -- i.e., a belief that mathematics really does make sense and really can be discovered?

We shall explore this last possibility in the next two examples.

Example 6. Can Mathematics Be Discovered? Kye's Arithmetic. (The Problem of the Unexpected Response.) In a third grade class of academically superior (upper 33% of school population in a good suburban school) boys and girls, the teacher had studied with a teacher who had studied with the Madison Project. This might then be called a class on the periphery of the Project's work.

The teacher was explaining how to subtract with "borrowing" or "re-grouping." She was using the example

$$\begin{array}{r} 64 \\ \underline{-\ 28} \end{array}$$

and was explaining "you can't subtract 8 from 4, so you must regroup the 60 as ..." At this point Kye, a third-grade boy, interrupted and said:

"Oh, yes, you can! Four minus eight is negative four

$$\begin{array}{r} 64 \\ \underline{-\ 28} \\ {}^{-}4 \end{array}$$

and 60 minus 20 is 40

$$\begin{array}{r} 64 \\ \underline{-28} \\ {}^{-}4 \\ 40 \end{array}$$

and so the answer is 36"

$$\begin{array}{r} 64 \\ \underline{-28} \\ {}^{-}4 \\ \underline{40} \\ 36 \end{array} .$$

This has struck Project teachers as one of the best results they have ever achieved (and one not often equalled).

Remark 1. Kye could not have done this if he had not previously studied the arithmetic of signed numbers in grade 2.

Remark 2. Neither the teacher, nor anyone else associated with the Madison Project, had ever seen this algorithm before. It was, to us as well as to Kye, a brand new discovery made by a third grade boy.[21]

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[21] We have since seen it in several places, and several other teachers have reported their students making this same discovery. Cf., e.g., Sandel (83). See also Cambridge Conference on School Mathematics (9). Kye's discovery was made in 1961. A film shows Kye and his classmates working with an elaboration of his theory (the original discovery, obviously, occurred in a class which was not being videotaped). Cf. Davis (23), pp. 74-76.

Remark 3. This episode is almost the epitome of the distinction between what the Project considers a "modern" mathematics program as against a "traditional" program.[22] The teacher *listened* to Kye, and *thought about* what Kye was saying. A "traditional" teacher -- or, in Sawyer's language, a "poor" teacher -- would have said something like "No, Kye, that's not how you do it! Now please pay attention, don't interrupt, and I'll show you again how it goes." Such a "traditional" response would have left Kye with the feeling that "this math never does make sense; just when you think you've gotten it figured out, the teacher tells you you're all wrong. It doesn't pay to try to figure it out. It's better to wait and listen to how the teacher tells you to do it." A consequence of "not trying to figure it out" would then be a behavior commonly enough observed in students: an inability to judge whether their work is correct until the teacher tells them whether they are right or wrong.

Remark 4. From the *teacher's* point of view, notice how *completely unexpected* a student response can be, *and yet be entirely correct*.

Remark 5. From instances such as this, it seems reasonable to suppose that schools *do* teach people either that *mathematics can be discovered*, or that it cannot be; either that *they* can discover mathematics, or else that they cannot.

Remark 6. The Project's collection of recorded lessons reveal that correct but entirely unanticipated student responses occur very often -- quite likely in most classes at least once a day, or thereabouts. Nearly all of them go unrecognized by the teacher. In the next example we give another one that almost passed unrecognized.

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[22] It is conceded, of course, that "modern programs" are not necessarily *new*. Some teachers taught "modern programs" decades ago, and probably centuries ago. Warwick Sawyer correctly points out that the distinction between "good" and "bad" mathematics would be more accurate.

Example 7. The Problem of the Unexpected Response. Nancy's Matrix. In a ninth-grade film-recorded class entitled *Introduction to Infinite Sequences, Part I* (and listed in Appendix B as number 111), the teacher, intent upon an analytic construction of the real numbers via bounded monotonic sequences, was confronted with a girl named Nancy (who rarely spoke during mathematics classes) who said she could find a square root of two by using an isomorphism with matrices. The teacher nearly dismissed this as impossible since it appeared (at first glance) to replace the theory of limits by the theory of 2-by-2 matrices, an unlikely line of attack, and also because Nancy had never previously made a valid original contribution in class. Fortunately, the teacher had recently had a conversation with Professor Andrew Gleason of Harvard University, on precisely this subject, and with this in mind was able to guess what Nancy proposed to do -- hence the teacher allowed her to do it, and she did.

The recognition of correct student answers in mathematics is an exceedingly difficult task. Yet somehow providing for this must be one of the fundamental goals of programs for the improvement of school mathematics.[23]

Example 8. What Good Is the Commutative Law for Addition? There is a tendency in "new math" to place considerable stress on the *fact* that

$$3 + 4 = 4 + 3 .$$

Within Madison Project experience, students do not find this interesting. From the Project point of view, the students are correct. This *fact* -- regarded as a curiosity -- is *not* interesting.

The fact that this "always works" -- i.e.,

$$\forall_x \ \forall_y \qquad x + y = y + x \text{ --}$$

---

[23] On this issue of the openness of the school to student originality (or even to student dissent), cf. Schrag (86), where this appears as a central theme in assessing the effectiveness of the school -- perhaps as *the* central theme.

is possibly more interesting, but not much so, *if we really regard it as an isolated fact*.

Mathematicians find this identity valuable -- and even exciting -- precisely because they do not regard it as an isolated fact. They use it in the *process* of solving problems, as in

$$\begin{aligned} 18 + 32 &= (10 + 8) + (30 + 2) \\ &= (10 + 8) + (2 + 30) \quad \Big\rbrace\ \text{CLA} \\ &= 10 + [8 + (2 + 30)] \\ &= 10 + [(8 + 2) + 30] \\ &= 10 + [10 + 30] \\ &= 10 + 40 \\ &= 50 \quad , \end{aligned}$$

which is *somewhat* more interesting, but hardly *interesting*.

The real excitement comes from the *process of describing a mathematical system by means of axioms*. That this process is possible at all is exceedingly remarkable, and its use is one of the things that has added greatly to the excitement of mathematics in recent decades.

The Madison Project gives students experience with this process in the following way:

First, students learn to discriminate between open sentences which can be proved *not* to be identities, vs. those which cannot be proved not to be (for the moment).

Secondly, using this decision procedure, a tentative list of "identities" is compiled.

Third, this list is made as long as possible.

Fourth, methods for "shortening this list without losing anything" are sought, and generalization and implication are obtained as two such methods.

Fifth, using these methods, the list is "shortened as much as possible without losing anything."

Sixth, as a result, axioms are distinguished from theorems.

With younger students (e.g., fifth graders), the process stops here.[24] With older students (e.g., ninth graders), the process continues to further steps:

Seventh, the logical processes that have been used more or less implicitly are now exposed to explicit study.[25]

Still further steps are possible: one can generalize the logic itself; one can formulate the logic more explicitly and then go back over the work with algebraic identities to see if it is still acceptable in the face of the new formulation of logic; and so on.

Of course, the almost inevitable next steps in this process are to change the axioms and see what mathematical structure emerges, or else to start with a different mathematical structure given in implicit or intuitive form, and to devise for it an appropriate explicit axiomatic description.[26]

The point, of course, is that all of these activities are processes to be carried out, and not facts to be learned.

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[24] A typical fifth grade tape-recorded lesson is available on the tape numbered D-1, and entitled: "□ + □ = 2 x □."

[25] Cf. Whitney (110), for one of the finest discussions of the process approach to the study of mathematics. Polanyi's distinction between the unexamined use of language vs. the examined use of language is probably one of the basic principles for organizing curriculum and instruction (cf. Polanyi (77)).

[26] Cf., e.g., Steiner (98).

Example 9. What Kind of Geometry Belongs in the Elementary School Program? One curriculum selection question which is by no means settled at the present time is: what kind of geometry belongs in the elementary school program? One could propose many alternatives: the study of symmetry and translations using physical materials, via the laws of optics, as in Marion Walter's "Mirror Cards";[27] the study of rudimentary aspects of analytic (Cartesian) geometry; the study of Euclidean constructions (as developed, for example, by Suppes and Hawley);[28] the learning of definitions of terms such as "ray," "segment," "line," etc., in a basically Euclidean framework; the study of such "topological" matters as continuity, and the use of these ideas in studying shadow figures on the wall, etc.; the study of visual perception, optics, and elementary aspects of projective geometry; the action of "taking giant steps" and the formalization of this into geometry based on vector algebra; the use of manipulation of symmetric objects (such as cardboard triangles) in order to generate binary operation tables for "followed by" on the set of allowable motions (some would say this is algebra rather than geometry); learning a vocabulary of three-dimensional forms, such as "cylinder," "cone," "sphere," "ellipsoid," "hemisphere," etc.; the use of topological ideas such as continuity, monotonicity, fixed points, etc., in the study of functions (as David Page has done with great success); and so on.

As can be inferred from the earlier examples, the Madison Project would opt for those kinds of geometrical studies which are oriented toward doing something, and would try to minimize those geometrical studies which lean toward -- or which, through degradation, will gradually come to lean toward -- merely memorizing facts.

In its own case, the Madison Project has selected mainly Cartesian analytic geometry, since this serves immediately as a vehicle for establishing close relations between arithmetic, algebra, geometry, physical science, social science, statistics, and probability, in ways which are meaningful to quite young children, and in ways that lead naturally toward many active processes.[29] Nonetheless, the Project recognizes the value of many of the alternative approaches (and in fact makes use of many of them), *provided they tend toward processes rather than toward the memorization of names or of "facts."* (For example, the Project uses Marion Walter's "Mirror Cards," and also uses *geoboards* (or "nail-boards"), as described in Cohen (14).)

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[27] Walter (108) and (107).

[28] Suppes and Hawley (102).

Example 10. "Show Me What People Mean by 3 Plus 5." Some of the examples given here may seem unrealistically exotic,[30] but it is our purpose to show that mathematics, from the early explorations of a four-year-old to the research work of professional mathematicians must always and necessarily contain a major ingredient of *process*.

The present example is on the elementary side.

A lesson which the Project uses with first or second grade children (or even younger children) consists of giving the children attractively-colored plastic washers, and giving them the task: "Show me what people mean by 'three plus five'." Children make various arrays, such as

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[29] Cf., e.g., the film-recorded lessons entitled: *Postman Stories*, *Circles and Parabolas*, *Graphing a Parabola*, *First Lesson*, *Second Lesson*, *A Lesson with Second-Graders*, *Graphing an Ellipse*, *Introduction to the Complex Plane*, *Weights and Springs*, *Experience with Linear Graphs*, *Graphs and Truth Sets*, *Tic-Tac-Toe in Four Quadrants*, and *Small-Group Instruction: Committee Report on Rational Approximations*. (See Appendix B.)

[30] But if the reader is inclined to dismiss all of this, he should first consider the large amount of work which the Project has done with thousands of children, *including culturally deprived children* and including extremely young children. The *process* of using the Marion Walter mirror cards has been carried out easily (and seemingly naturally) with children as young as three years and six months old, and is quite easy and natural for five-year-olds.

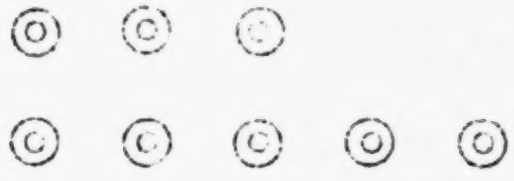

Figure 18. A child's response to "Show me what people mean by 'three plus five'."

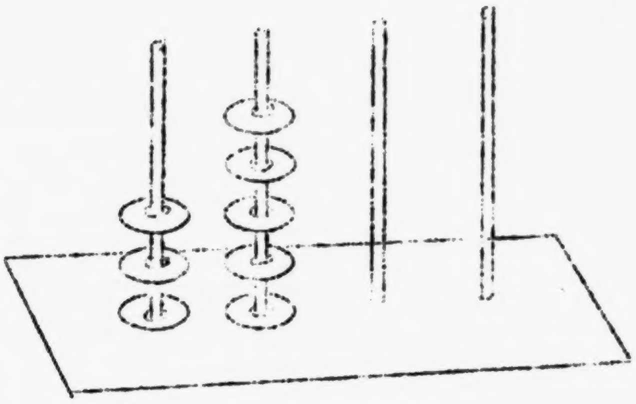

Figure 19. Another child's response to "Show me what people mean by 'three plus five'."

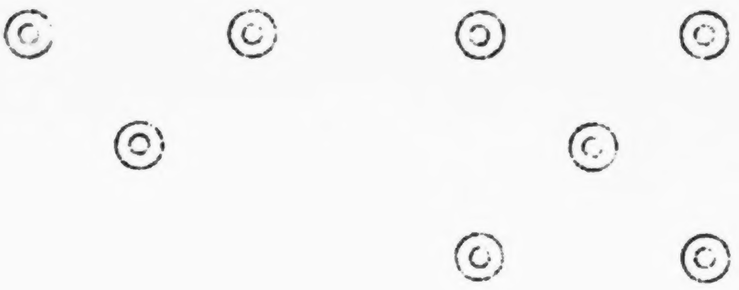

Figure 20. Yet another child's response to "Show me what people mean by 'three plus five'."

What is most important is that children focus on the *process* of arranging the plastic washers in order to "show the teacher," as requested. *They do not focus on the answer; yet once the washers are arrayed, every child can easily state the answer:*

$$3 + 5 = 8 .$$

Two lessons of this type have been recorded on film, and can be viewed essentially as they occurred; they are on the films entitled *Addition and Multiplication Using Plastic Washers* and *Multiplication Using Dots*.[31]

Example 11. Should "Sets" and "One-to-One Correspondences" Appear in the Elementary School Curriculum? By now it should be clear that the Madison Project would tend to decide this by asking: will the introduction of "sets" and "one-to-one correspondences" tend toward the *memorizing of facts*, or will it tend toward the *processes* of explication, problem-solving, model building, etc., *in which the students will be playing an active role?*

The Project's present tentative answer is to relate this to the basic *process* of *re-formulating*, *re-defining*, and *extending systems*, which will be discussed in a later section of this report. Obviously, a Project devoted to a "process" approach can announce only *tentative* "answers," but the tentative answer would be to *avoid* the use of the idea of "set" in analyzing primary grade mathematics.

For the present, the point is that this decision would be settled primarily in relation to the "fact" vs. "process" distinction for what goes on in the classroom.

An *alternative* "active" or "process" use of these ideas will be presented in the next (and final) example. Note that the Madison Project answer would tend to be: *do not introduce an idea if you do not intend to USE it!*[32]

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[31] Compare also the chapter "A First Lesson in Arithmetic with a First Grade in South Texas," in Cochran (12), which has been included in the present report as Appendix E.

Example 12. The Process of Solving One Problem from Professor Wylie's Book. A paperback volume that deserves serious consideration for study by children where possible, and in any event by teachers, in order that they may gain further experience with the *process* of mathematics is: C. R. Wylie, Jr. *101 Puzzles in Thought and Logic* (111).

The first of Professor Wylie's 101 problems is:

"In a certain bank the positions of cashier, manager, and teller are held by Brown, Jones and Smith, though not necessarily respectively.

"The teller, who was an only child, earns the least.

"Smith, who married Brown's sister, earns more than the manager.

"What position does each man fill?"[33]

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[32] This is, of course, not a new notion in western thought, but it *is* contrary to the common "traditional" practice of most schools. (Note that "using" an idea means "using it in a significant, interesting, challenging, and possibly even exciting way." Routine "uses" such as drill, etc., do not ordinarily justify the introduction of the idea.) Cf. Alfred North Whitehead's comments, made prior to 1929 (Whitehead (109); page references given here are to the paperback edition of July, 1949):

"In training a child to activity of thought, above all things we must beware of what I will call 'inert ideas' — that is to say, ideas that are merely received into the mind without being utilised, or tested, or thrown into fresh combinations.
(continued on page 36)

[33] From: *101 Puzzles in Thought and Logic* by C. R. Wylie, Jr., Dover Publications, Inc., New York, 1957. Reprinted through permission of the publisher.

We now consider the process of solving this problem.[34]

We may "model" the problem by imaging two sets of names.[35]

$$\{\text{cashier, manager, teller}\}$$

and

$$\{\text{Smith, Jones, Brown}\} \ ,$$

and our task is to establish a "one-to-one" correspondence between these two sets which will reflect the conditions of Professor Wylie's problem.

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[32] (continued)

"In the history of education, the most striking phenomenon is that schools of learning, which at one epoch are alive with a ferment of genius, in a succeeding generation exhibit merely pedantry and routine. The reason is, that they are overladen with inert ideas. Education with inert ideas is not only useless: it is, above all things, harmful -- Corruptio optimi, pessima. Except at rare intervals of intellectual ferment, education in the past has been radically infected with inert ideas. That is the reason why uneducated clever women,

(continued on page 37)

[34] The reader who is interested in the "practical" use of problems of this type -- or, rather, processes of this type -- may wish to refer to: Berger, Cohen, Snell, and Zelditch (3); and also to the well-known volume Kemeny, Snell, and Thompson (61).

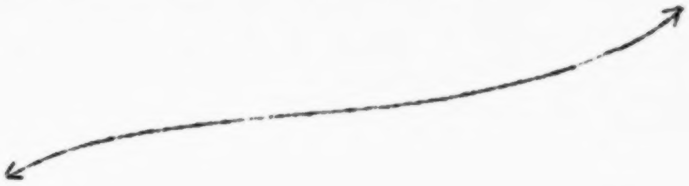
[35] Notice that we could not interpret these as "sets of people," for in such a case the two sets would be identical and the desired mapping would not have the desired meaning. This may serve as a caution to anyone who is over-enthusiastic about "simplifications" resulting from the use of sets.

Actually, we do not need set notation at all; we can settle for a two-column array,

$$\begin{array}{ll} C & S \\ M & J \\ T & B \quad , \end{array}$$

and our task is to draw lines connecting these symbols in the proper fashion.

What is required is, in a way, the invention of two symbols, which we shall (somewhat arbitrarily) choose as



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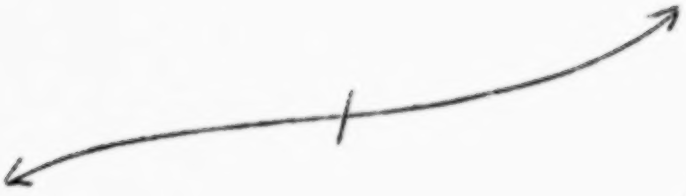
[32] (continued)
who have seen much of the world, are in middle life so much the most cultured part of the community. They have been saved from this horrible burden of inert ideas. Every intellectual revolution which has ever stirred humanity into greatness has been a passionate protest against inert ideas. Then, alas, with pathetic ignorance of human psychology, it has proceeded by some educational scheme to bind humanity afresh with inert ideas of its own fashioning.

"Let us now ask how in our system of education we are to guard against this mental dryrot. We enunciate two educational commandments, 'Do not teach too many subjects,' and again, 'What you teach, teach thoroughly.'

"The result of teaching small parts of a large number of subjects is the passive reception of disconnected ideas, not illumined with any spark of vitality. Let the main ideas which are introduced into a child's education be few and important, and let them be thrown into every combination possible. The child should make them his own, and should understand their application
(continued on page 38)

and



,

where the cross-mark on the second curved line indicates negation.

We proceed at an intuitive, heuristic level, and leave the reader to carry out the process of achieving a more formal exposition if he desires.

i) "The teller is an only child," and "Brown has a sister," so we write:

$$\begin{tikzcd} C & S \\ M & J \\ T \arrow[r, leftrightarrow, "/" description] & B \end{tikzcd}$$

---

32 (continued)
here and now in the circumstances of his *actual life. From the* very beginning of his education, the child should experience the joy of discovery. The discovery which he has to make, is that general ideas give an understanding of that stream of events which pours through his life, which is his life.

. . .

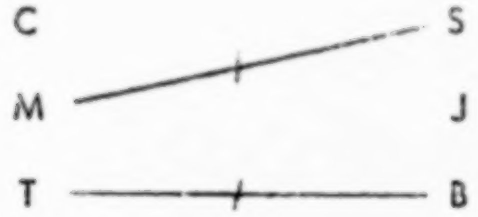
"Passing now to the scientific and logical side of education, we remember that here also ideas which are not utilised are positively harmful.

. . .

"Education is the acquisition of the art of the utilisation

(continued on page 39)

ii) "Smith earns more than the manager," so we write



iii) "Smith earns more than the manager," and "the teller earns the least." Therefore Smith cannot be the teller, and we write:

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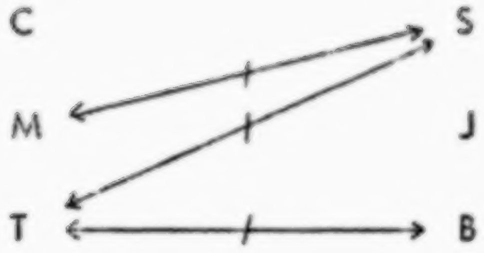
32 (continued)
of knowledge. This is an art very difficult to impart. Whenever a textbook is written of real educational worth, you may be quite certain that some reviewer will say that it will be difficult to teach from it. Of course it will be difficult to teach from it. If it were easy, the book ought to be burned; for it cannot be educational.

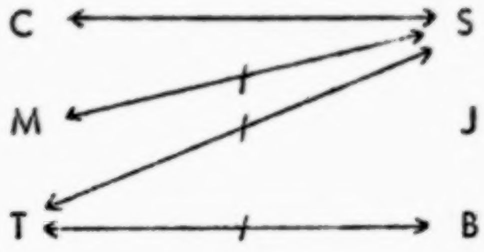
...

"We now return to my previous point, that theoretical ideas should always find important applications within the pupil's curriculum. This is not an easy doctrine to apply, but a very hard one. It contains within itself the problem of keeping knowledge alive, of preventing it from becoming inert, which is the central problem of all education." (From: The Aims of Education by Alfred North Whitehead (109), pp. 13-17. Copyright, 1929, by The Macmillan Company. Reprinted by permission of The Macmillan Company.)

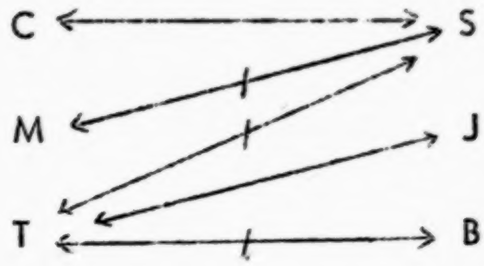
Contrast this with Peter Schrag's observation on contemporary schools in the United States: "What is going on in the classroom, and why? Is the tired textbook, for example, merely an archaic device sustained by pedants, or is it also an effective instrument protecting the community from the teacher's incompetence and the teacher from the community's prejudices -- a warranty of acceptable idealogical practice?" (From: Voices in the Classroom by Peter Schrag (86), p. 6. Copyright © 1965 by Peter Schrag. Reprinted by permission of the publisher, Beacon Press.)



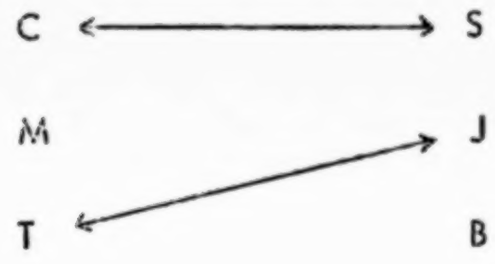
iv) We need no more references to Wylie's verbal problem. Everything else that we need can be read from our diagram. Since "S" is not matched with either "M" or "T," it must be matched with "C," and therefore we write:



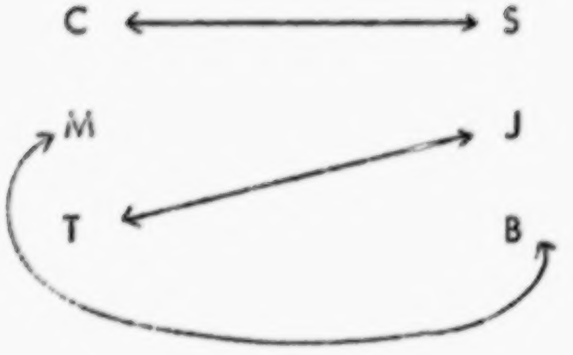
v) Since "T" cannot be matched with either "S" or "B," it must be matched with "J," and we write:



vi) Finally, since we have



we must also have



We can now announce the solution: Smith is the cashier; Jones is the teller; and Brown is the manager.

Which was more important: the "answer," or the process of obtaining the answer?

(Notice that this qualifies as a "process" by virtue of the fact that no one had told us in advance how to go about solving the problem, so that a measure of original invention was necessarily involved.)

We have argued that one of the most serious weaknesses in education is the seemingly persistent tendency for a curriculum of processes to degenerate into a curriculum of facts. The problem is not new, and was identified by, Alfred North Whitehead in terms quite similar to those used in the present argument. Whitehead goes further -- as we would, also -- to argue that one of the highest priorities in education must be accorded the task of fighting against this degradation from "living process" to "inert ideas."

The American classroom today generally loses this battle; ample evidence is available from observations by Louis Smith, Peter Schrag, and others.[36]

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[36] Cf. Smith and Geoffrey (97); Schrag (86); also Mayer (72).

Notice the contrast between Schrag's description of a laboratory in a school in Kansas:

> "Topeka High has made few concessions to the ultramodern; many of its teachers follow conventional textbooks, impart information when they can, ask direct questions that can be answered by finding the right page or the right set of notes, and conduct laboratories by the recipe method. One of the most popular teachers says he runs his science classes by 'giving them instructions. They have to listen. If they make mistakes they come back after school. I tell them that if they don't like the grades I give them on lab exercise they can come back and do some more work.' His cosmos leaves little room for originality, and the grade is -- as they say in the trade -- negative reinforcement imposed according to the number of errors the student makes in mixing up the recipe."[37]

as against Rosenthal's proposal for how laboratory work in psychology (and elsewhere) should be conducted (but, of course, is not at present):

> "Whereas most instructors of laboratory courses in various disciplines tend to be very conscious of experimental procedures, students tend to show more outcome-consciousness than procedure-consciousness. That is, they are more interested in the data they obtain than in what they did to obtain those data. Perhaps the current system of academic reward for obtaining the "proper" data reinforces this outcome-consciousness, and perhaps it could be changed somewhat. The selection of laboratory experiments might be such that interspersed with the usual, fairly obvious demonstrations there would be some simple procedures that demonstrate phenomena that are not well understood and are not highly reliable. Even for students who "read ahead" in their texts it would be difficult to

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[37] From: Voices in the Classroom by Peter Schrag (86), p.18. Copyright © 1965 by Peter Schrag. Reprinted by permission of the publisher, Beacon Press.

determine what the 'right' outcome should be. Academic emphasis for all the exercises should be on the procedures rather than on the results. What the student needs to learn is, not that learning curves descend, but how to set up a demonstration of learning phenomena, how to observe the events carefully, record them accurately, report them thoroughly, and interpret them sensibly and in some cases even creatively.

"A general strategy might be to have all experiments performed before the topics they are designed to illustrate are taken up in class. The spirit, consistent with that endorsed by Bakan (1965),[38] would be 'What happens if we do thus-and-so' rather than 'Now please demonstrate what has been shown to be true.' The procedures would have to be spelled out very explicitly for students, and generally this is already done. Not having been told what to expect and not being graded for getting 'good' data, students might be more carefully observant, attending to the phenomena before them without the single set which would restrict their perceptual field to those few events that illustrate a particular point. It is not inconceivable that under such less restrictive conditions, some students would observe phenomena that have not been observed before. That is unlikely, of course, if they record only that the rat turned right six times in ten trials. Observational skills may sharpen, and especially so if the instructor rewards with praise the careful observation and recording of the organism's response. The results of a laboratory demonstration experiment are not new or exciting to the instructor, but there is no reason why they cannot be for the student. The day may even come when classic demonstration experiments are not used at all in laboratory courses, and then it need not be dull even for the instructor. That

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[38] Rosenthal's reference here is to: D. Bakan, "The mystery-mastery complex in contemporary psychology. Amer. Psychologist, 1965, 20, 136-191.

> the day may really come soon is suggested by the fact that so many excellent teachers are already requiring that at least one of the scheduled experiments be completely original with the student. That, of course, is more like Science, less like Science-Fair."[39]

Our proposal to shift the learning emphasis to process is not unique to the Madison Project. Essentially this same proposal has been made (as we have just seen) by the psychologist Rosenthal, but also by the biologists J. J. Schwab[40] and Edwin B. Kurtz, Jr.,[41] by the mathematician Hassler Whitney[42] (and also by E. E. Moise, R. L. Moore, George Polya, and others), by the writer Hughes Mearns,[43] by the science educator Mary Lela Sherburne,[44] by the British mathematics educator E. E. Biggs,[45] by the eminent German mathematics educator Hans-Georg Steiner,[46] and, as we have seen, by Alfred North Whitehead. Many other scientists and educators could be cited in support of such a shift of attention (including J. Richard Suchman, David Page, Carl Rogers, Earl Kelley, Donald Snygg, and Richard Feynman, among others).

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[39] From: Experimenter Effects in Behavioral Research by Robert Rosenthal (82), pp. 35-36. Copyright © 1966 by Meredith Publishing Company. Reprinted by permission of Appleton-Century Crofts.

[40] Schwab (90).

[41] Kurtz (65).

[42] Whitney (110).

[43] Mearns (73).

[44] Both in the film Classrooms in Transition and in the report The Cardoza Model School District: A Peach Tree Grows on T Street (92).

[45] Cf. the film I Do...And I Understand, available from Mr. S. Titheradge, Manager, New Print Department, Sound Services, Ltd., Wilton Crescent, Merton Park, London, S.W. 19, England.

[46] Steiner (98).

Nonetheless, many "new math" programs are just about as much "fact-oriented" as the average traditional program was. Progress on this front has proved elusive.

The difference between the role of the textbook in a college physics course, as against a second-grade arithmetic course, must be emphasized. Good university texts are, indeed, "process oriented," if only because of their emphasis on solving interesting and varied problems. The "tired textbook" Schrag finds at the pre-college level (and which is especially noticeable in the elementary school) is quite a different matter, and Schrag is probably correct in suspecting that it is a symptom of some deeper underlying problem.

It must further be emphasized that a "process-oriented" approach is *not* less accessible to children; quite the contrary: children prefer it, and flourish under it, *as long as the school can measure up to the severe demands of providing a suitable learning environment.*

If a single major purpose were to be stated for the work reported here, it would be: to decrease the present "fact emphasis" of the schools' curriculum and instruction in mathematics, and to replace it by a livelier, more rewarding *process* orientation, which would have the further advantage of rendering a truer picture of what mathematics is all about.

B. "Learning in Context"

This same purpose can be stated from a different point of view: it is a major purpose of the Project to enable children to learn new ideas *in a context where they will be using these ideas as tools.* There is considerable evidence that our purposes shape our perceptions, and probably also our associations, so that an "idea" is *not* independent of the context in which it is learned.

C. Restatement of the Goal of This Project

The Project described in this report was not concerned with the production of textbooks. It was concerned with the actual encounters with mathematics which actual children would have in actual classrooms. The changes in the nature of these encounters which the Project has sought to bring about could be defined narrowly as a switch from fact-oriented encounters to process-oriented encounters. It is important to keep in mind that this goal is really a narrow corollary to a broader goal -- namely, the goal of making these experiences as pleasurable and profitable as possible in the eyes of the children, and as "mathematical" as possible in the eyes of mathematically-competent professionals.

It should further be emphasized that these goals have been pursued with culturally-privileged children, with "ordinary" children, and with culturally-deprived children. In every case, the mathematician and the child -- whatever his socio-economic background -- have usually both preferred the process-oriented approach to a fact-oriented approach. The task has been to devise suitable "process" experiences, and to give teachers the necessary educational background to cope with the demands of this style of teaching.

The results have been recorded on audio-tape, video-tape, and film. This method of recording has been made possible by financial support from the Marcel Holzer Foundation, the Alfred P. Sloan Foundation, by Webster College, and by the National Science Foundation.

II. *Method*

The method to be used is rather clearly dictated by the purpose. Since the purpose is to shift from "mathematics as the memorizing of facts" to "mathematics as the *processes* of explication, invention, creation, description, analysis, etc.," it is clear that *the point of intervention had to be the actual classroom experiences of children.* It was not judged feasible, when this Project began eleven years ago, to intervene in the educational process at the level of preparing sample textbooks. It was also not judged feasible to intervene at the level of teacher education, since one would not have known what to "teach" teachers in order that *they* might themselves teach a curriculum that did not exist within an instructional milieu with which no one was familiar.

Thus the point of intervention in the educational process was chosen to be the child's actual experiences in the classroom.

If this is where the "result" is to be applied, one can still ask where the creation of the result is to occur. Again, alternatives exist: one could pre-plan a "process" curriculum during a summer writing session, to give one example. Here, also, the decision -- based on the world of 1960 or thereabouts -- was that *the creation should itself occur in classrooms.* It seemed doubtful that "lesson plans" devised entirely by adults -- during, say, a summer writing session -- would turn out to be viable in actual classroom usage. Hence the procedure was developed of having a team of mathematically-competent people and educationally-sophisticated people work out a flexible and tentative lesson plan, try it out with children, discuss it, revise it and polish it, subject it to further trials, and -- when it seemed to be in reasonably stable shape -- *teach it to children to whom it was a new lesson*, and *record the interplay between teacher and children on film, videotape, or audiotape*. The ultimate results were of two types: the "impersonal" or "concrete" result was the set of films listed in Appendix B, plus a larger collection of audio-tapes some of which are listed in Appendix C. The "human" or less tangible result was the creation of a group of teachers who knew how to let children experience "mathematics as a process," -- and, in fact, the simultaneous creation of children who had participated in such a program.

It was envisioned in the original U.S.O.E. project that such teachers would number several dozen, or possibly a few hundred. This objective was modified when Knowles Dougherty, Dr. Samuel Shepard, Ogie Wilkerson, Katherine Vaughn, Katie Reynolds, and others, in the Banneker District of St. Louis, established clearly that this "process" approach was every bit as viable with culturally-deprived children in urban slums as it was with suburban (or even highly-privileged) children, and when, subsequently, Mrs. Evelyn Carlson, Associate Superintendent of Schools in Charge of Curriculum Development, Chicago, Illinois, requested the Madison Project to set up a program to provide Chicago Public School teachers with the necessary educational background to use these same materials and procedures in something of the order of 12,000 classrooms in Chicago. The program subsequently expanded to include also teachers in New York City, Philadelphia, Los Angeles, and San Diego County. Some of this extension has been financed by San Diego, Los Angeles, Chicago, and Philadelphia, and some of it has been financed by the National Science Foundation. All of it is an outgrowth of the original U.S.O.E. program.

There are thus four products:

i) "process" lessons

ii) recordings of these on film or tape

iii) teachers capable of teaching these lessons

iv) children who have been participating in this program.

Furthermore, the original U.S.O.E. Project, aimed at dozens (or possibly hundreds) of suburban teachers and their students, came to blend smoothly into a much larger Project, sponsored by various cities, counties, and by the National Science Foundation, which deals with urban children and with their teachers in five major urban areas of the United States (Chicago, San Diego, Los Angeles, New York City, and Philadelphia), plus smaller programs in St. Louis and in Washington, D. C.

A. The Curriculum Aspect

To return to the "process" lessons themselves, one can to some extent identify a "curriculum aspect" and an "instructional aspect" (or "milieu aspect").

Thusfar, little has been said of the curriculum aspect. Even if mathematics is to be experienced as a process, there remains some question of what mathematics should become the focus of these experiences.

One of the best discussions of the curriculum realm has been given by Jack A. Easley, Jr.[47] Easley (as well as David Hawkins, and others) views the realm of possible mathematical experiences -- from the point of view of "curriculum" or "content," the "what" rather than the "how" -- as a large domain, largely unexplored. One could suggest the analogy of Lewis and Clark exploring the northwest, but even that is not fully accurate -- since, in order to have something to explore, one must create the relevant experiences -- the trees, mountains, and rivers were already there for Lewis and Clark to discover. It is not so with appropriate experiences in statistics, logic, or vector analysis for ten year olds. Suitable experiences must be created (from a teacher's viewpoint), before they can be experienced (from the child's point of view). A closer analogy might be to Beethoven, "exploring" a world of possible musical experiences that could only be experienced after they had been imagined and created.

How can a project organize this "topographic exploration" of a domain that must be created before it can be explored?

The Madison Project's answer has consisted of several parts:

i) In the first place, the Project recognized that it lacked the resources to create a complete curriculum. Rather than carry over much unsuitable material into a "new program" -- merely for lack of opportunity to find anything better -- the Project approached the curriculum as one might approach urban renewal. Most of the city was left untouched. Only in spots, where it was possible to make definite improvements, was the curriculum tampered with. This is sometimes expressed by saying that the Project's materials are "supplementary" to existing programs, but there is danger here in that the word "supplementary" has different meanings to different readers.

A school making use of the Project's materials would maintain their existing program intact, weave in Project materials where feasible, and eliminate portions of the curriculum -- either "new" portions or "traditional" portions -- as they become unnecessary or as they were proved to serve no purpose in the newly evolving fabric.

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[47] Easley (32).

ii) The "topic-extension" approach. The actual construction of curriculum materials has been based upon a "topic-extension" approach. A topic -- such as the concept of a mathematical variable -- is identified as being of very high priority. (Indeed, one can do very little modern mathematics without the concept of variable.) This serves as the focal-point topic from which one now builds. In working with children in the classroom, specialist teachers try alternative methods of letting children work with variables -- seeking always either processes that explicate the concept of variable itself, or else processes that should precede such explication, or else processes for subsequent use that employ the concept in significant further development (and not merely in repetitious "drill" exercises).

In the course of doing this, various other topics will appear which turn out to be more-or-less inextricably intertwined with the classroom work with the original topic.

For example, consider the case already mentioned: the use of variable.

To begin with, there are a wide variety of ideas presently in use that more or less resemble "variable," so a selection must be made among them: there is the issue of the rule for substitution that says: "whatever replacement is made for the first occurrence of a variable, that same replacement must be made in every occurrence of that variable." This rule is always used in ordinary mathematics and in mathematical logic. Sometimes it is stated very explicitly, sometimes it is left tacit. The Project's decision was: since this is a pure convention (that is to say, an essentially arbitrary agreement among people) and not a "fact of nature," it cannot possibly have the status of a "self-evident truth," nor can it possibly be "discovered" in any scientific sense of the word discovery. Therefore it must be dealt with explicitly, and this treatment has been adopted.

Some present "modern" textbooks do not use this agreement at all. They allow, for example,

$$\square + \square = 5$$

to stand for such addition facts as

$$3 + 2 = 5.$$

It appears to the Project that such a use, being entirely at variance with mathematical practice, should be avoided, and the Project does avoid it.

There are other variations in the meaning of the concept of variable. Sometimes it merely indicates an incompleted portion of a mathematical statement; sometimes it is a letter used to represent a specific number (which is often the use in

$$3 + 2 = \square,$$

where "$\square$" is assumed to be a name for 5). Following virtually universal practice among mathematicians and logicians, the Project rejects all of these deviant interpretations.

There is the Newtonian concept of "that which varies" vs. the modern logician's concept embodied in Beberman's word pronumeral, which involves a defined replacement set, and so forth.

The Project uses the Beberman approach, but adjusts the level of explicit verbal refinement to the child: sometimes "replacement set" is not explicitly considered, and sometimes it is, depending upon the experience and sophistication of the children.

There are, then, various decisions that the teacher must make (or the curriculum-developer must make) as to just how the concept is to be defined.[48]

After the concept of variable begins to assume some clarity, there is the question: what other concepts are inextricably related to classroom "process" experiences involving variables?

One such concept, clearly, is the extension to open sentences, truth sets, and open sentences involving more than one variable. These ideas must then be developed, in order to cope adequately with the original concept of variable itself.

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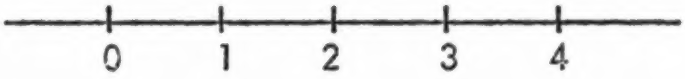
[48] Even the discussion presented here is sharply abridged, and omits many questions, such as distinctions between "variables," "constants," "parameters," etc.

Furthermore, once one introduces truth sets, it is necessary to introduce methods for representing truth sets.[49] This leads almost inevitably to Cartesian coordinates, graphs, and infinite sets.

This, in turn, leads virtually inevitably to a consideration of signed numbers (i.e., integers or rational numbers, positive, negative, or zero). The extension would be hard to avoid, and any attempt to avoid it would introduce artificial distortions into the curriculum. Here are some of the reasons:

a) Once the child deals with a number line



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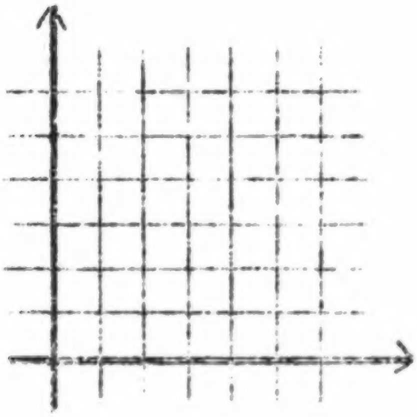
he will naturally ask about number names for *other* points on this line; this kind of inquiry is *precisely* the kind of "process" that we seek to foster; hence we have no choice but to pursue this inquiry with the child, and we are thus led to both fractions and negative numbers.

b) The same argument applies to Cartesian coordinates



once you can plot points in the first quadrant.

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[49] It is not, however, necessary to develop the concept of "set" itself; this can remain part of the tacit unexamined part of the structure *at this point*.

c) Once one has variables and open sentences, the child -- if the milieu encourages the process of "exploration" -- will inevitably come up with problems like

$$(\square \times \square) - (2 \times \square) = \triangle ,$$

which will in turn lead him to problems like

$$(1 \times 1) - (2 \times 1) = \underline{\quad\quad} .$$

Hence, at the very least, one must provide negative numbers as names for points on the number line, and as "answers" for questions such as

$$1 - 2 = {}^{-}1 .$$

iii) Comparison with larger bodies of knowledge. This example has already given some indication of how the topic-extension approach can be used by a specialist teacher trying out a concept area with children. This same example brings us to another criterion for curriculum development: as promising "pieces" of curriculum begin to take shape, one can compare them with larger bodies of knowledge to see whether they look as if they will fit well into the larger picture.

In the present case, we can compare this small-but-promising piece of mathematics curriculum (which the Project has tested in grades 2, 3, 4, 5, 6, 7, 8, and 9) with emerging science curricula in these areas. The result (in this case) confirms the appropriateness of this piece of curriculum, since nearly every elementary school science program finds it expedient to develop Cartesian coordinates. It appears that the emerging piece of mathematics curriculum will relate easily to modern elementary school science programs.[50]

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[50] The Project has gone further, and run its own trials on relating graphs to social studies. Again, graphs prove to be an exceedingly valuable tool -- for example: make a graph showing the population of your city at ten-year intervals, etc.

One can compare this small piece of proposed mathematics curriculum against advanced contemporary mathematics. This requires what Polanyi calls the art of proper identification, since the elementary notation

$$1 - 2 = {}^{-}1$$

could be *interpreted* in many ways. Some of these might be judged unsuitable (and involve such issues as the need, or lack of need, to distinguish between isomorphic systems). The Project chooses to interpret this as a second-grade representation corresponding to the advanced definition of negative one as an equivalence class of ordered pairs:

$${}^{-}1 \overset{\text{def}}{=} \left\{(1,2),\ (2,3),\ (3,4),\ (4,5),\ \cdots\right\},$$

and hence accepts the elementary notation

$$1 - 2 = {}^{-}1$$

as being suitable for gradual refinement into acceptable advanced modern practice.[51]

iv) Notice that the *topic-extension* development has not yet reached a natural boundary, for, once we *introduce* the signed numbers, it becomes almost inevitable to ask "how one does arithmetic with them," and we have not yet answered this question, nor even begun to.

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[51] This criterion may seem vacuous. It is not. When it is used, it results in rejecting many possible curriculum innovations. For example, the notation of "arrows" to indicate *lines* wastes a valuable notation which deserves to be saved for the more important task of indicating *orientation* -- i.e., "the positive direction" -- on a line. This becomes especially important, for example, in studying falling bodies in physics, where the choice of a positive direction varies from one author to another. Cf. Klein (62).

v) Retain the topic only if suitable experiences can be devised. Given the goal of emphasizing processes, it is clear that a topic under consideration will be retained only if suitable experiences can be devised. In the present example they can be. The game of Tic-Tac-Toe can be modified to teach second-grade children to plot points in Cartesian coordinates (and other games are also available). The game "Pebbles-in-the-Bag"[52] can be used to give concrete experience with problems such as

$$1 - 2 = {}^{-}1 .$$

vi) Try to articulate an over-all theory. As pieces of curriculum are created in this way, it seems to be essential to try to articulate an over-all theory to guide the emerging shape of the curriculum, and to relate "curriculum" to "instruction," from which it has been separated only artificially for purposes of short-range analysis.

The emerging theory has many aspects. We mention one now, and will consider more later.

How many "different number systems" will we have in the elementary grades? The Project presently answers "three," as follows:

a) The "counting" and "how many" numbers, which must therefore include zero:

$$\left\{0, 1, 2, 3, 4, \ldots\right\}$$

b) The "measuring" or "sharing" numbers:

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, 2\frac{1}{5}, 1965, 10.1, \ldots\right\}$$[53]

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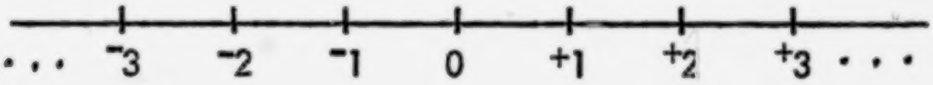
[52] Cf. the film-recorded lesson entitled: A Lesson with Second Graders.

[53] Some liberty has been taken here in the use of notation, in the interests of brevity; this set could be "listed," but the list given begins in an unpromising fashion for formal definition.

c) The "reference point" numbers, which arise in "reference point" situations (such as finding number names for the points on a line):



This system also includes $\frac{^{-}1}{2}$, $\frac{^{-}2}{3}$, $^{-}2\frac{1}{2}$, and so on.

This particular instance of an "over-all-theory" serves the function of tidying up the curricular housekeeping; it helps both teachers and children map reality into appropriate abstract models by suggesting general guidelines; e.g., "Is it [for some definite problem] a reference-point problem or a counting problem?"

This concludes the specifically curricular aspect of the discussion of methodology. We turn next to the relation between curriculum and instruction.

B. The Relation of Curriculum and Instruction

The example introduced already -- beginning with variable and "extending" as necessary to graphs, signed numbers, etc. -- can serve also to exhibit some of the methods used in relating curriculum aspects of school to instructional aspects.

i) Tacit or Explicit Introduction. As has already been mentioned, after the precise meaning for "variable" has been determined in relation to the curriculum plan, there is still the question of what experiences will the children have in order to gain their first acquaintance with this concept. In particular, in the case of variable, one can be rather fully explicit, or one can use an approach that is mainly tacit or implicit.

For most children the Project uses the tacit approach. The child is given a simple task such as:

teacher writes on board	teacher says
$3 + \square = 5$	"What can I write in the box in order to get a *false* statement?"
	[Children give various answers.]
	"What can I write in the box in order to get a *true* statement?"
	[Children answer "2".]

An *explicit* discussion of *variable* (including even the use of the word "variable") is deferred until later.

ii) *Notation.* The notation must be worked out in the classroom. Although the raised positive and negative signs were developed by David Page, Max Beberman, and the UICSM Project, it was *seventh-grade children* who pursuaded the Madison Project to adopt this notation. In preliminary classroom developments, problems such as

$$7 - 9 = -2$$

$$12 - 8 = +4$$

had introduced the integers (as discussed earlier). The question then inevitably arose: how do you do arithmetic with these new numbers? How do you add them, or multiply them?

The seventh-grade students -- since this was a *process* approach-- had the task of working out themselves an "arithmetic" for these numbers, and passing judgement on the suitability of their work. These seventh-graders easily determined addition and subtraction as usual, but for multiplication they chose

$$+2 \times -3 = +3$$

with the explanation

"two times three is six, and then you have to subtract three, so the answer is three."

The specialist teacher who was teaching the lesson (in order to develop appropriate curriculum materials) immediately switched to the Beberman-Page UICSM notation,

$$^{+}2 \times {}^{-}3 = \quad ,$$

explaining the words "positive" and "negative" and showing locations on the number line, and the students were able to go on to develop the usual arithmetic for the system of integers.

These are methodological and epistemological questions raised by any approach to curriculum design. In general, while these may remain troublesome theoretical questions, they are not obstacles to effective progress when curriculum and instruction materials are developed by work in the classroom. The vast amount of feedback data available to the specialist teacher who is doing his "creating" in a classroom, with students, is a good practical guarantee against going too far astray.

C. Viability Testing

After the specialist teacher believes he has some "process" lessons in reasonably stable form, the question arises: this stability has been tested with one teacher, and with (usually) from three to ten different classes, probably reasonably similar in socio-economic and educational background; how stable will these lessons be with a wider range of teachers, and with students of more diverse backgrounds?

If the specialist teacher believes the lessons may be appropriate for many teachers and many children, they are recorded on film, taught to more teachers, and taught by these new teachers to various classes, including culturally-deprived children, "ordinary" children, and culturally-privileged children. Various Project specialist teachers observe these classes, and determine whether the lessons appear to possess adequate stability in the hands of a larger number of teachers, and in use with children of varied backgrounds.

Examples of some lessons which have not survived these "viability" tests are given in Appendix F.

D. Possibilities of Extending the Curriculum

The result of developing more effective notations, and other similar devices, methods, and materials, is this: one is then able to extend the school mathematics curriculum in two directions. Obviously, one can extend toward "more" mathematics, the inclusion of new topics, and more sophistication. Various examples already make this clear, and it can be observed in many film-recorded lessons, such as Second Lesson; Graphing a Parabola; Guessing Functions; Average and Variance; Matrices; Solving Equations with Matrices; Axioms and Theorems; The Study of Functions -- Linear, Quadratic, and Exponential; Programming the IBM 1620, Using GOTRAN; Complex Numbers via Matrices; Graphing an Ellipse; Bounded Monotonic Sequences; and others.

What is interesting is that "extension" in another direction is also possible. The best of the new notations are clearer, the "process" approach is more inviting, and so one can try to extend school experiences also to children who did not previously pay much heed to mathematics. In doing this -- especially with culturally-deprived children -- it is obviously advisable to avoid verbal complexity, highly "explicit" formulations, and certain kinds of sophistication. Given this proviso, the extension of a "simpler" program to reach more children is entirely feasible, and the performance of culturally-deprived children is recorded on numerous films and tapes.

III. Results

An Introductory Note: As remarked earlier in this report, the work carried on by the Madison Project of Syracuse University and Webster College began in 1956 and continues at the present time. Over a hundred "specialist teachers," closely associated with the Project, have been involved, as have several thousand teachers less closely identified with the Project, and many thousands of children in their classrooms. If only in fairness to all those who have helped shape it, an undertaking of this size deserves to be reported as clearly as possible. The present report aims for such clarity; if the tone seems over-simplified, polemical, or even strident, this is not so much the result of deliberate intent as the inadvertent consequence of a striving for effective communication.

From the two-way communications of the past few years it has been clear that within the educational-academic community there have been those who have understood (and often even anticipated) the goals and methods of the Project; there are those who reject both; and there are those who are honestly puzzled.

Part of the difficulty may lie in disagreements about the nature of mathematics and science, or in disagreements about the nature of schools, or in disagreements about the nature of children. But perhaps the deepest part lies in a disagreement about the nature of knowledge itself. What does the teacher need to know? What does the curriculum planner need to know?

The affluence of our society extends to our libraries and our research activities, and here -- quite as much as in the world of more mundane commodities and services -- we must come increasingly to exercise choice. We cannot tell teachers all they need to know about teaching -- we must choose. Indeed, we must choose not merely content, but also the kind of content, and in fact even the media by which and form in which this "knowledge" is presented.

The "results" of the present project are not contained in this section, although a conventional view would decree that they should be. Indeed, the results are not contained in this report, nor in any other report. Yet it is no form of secrecy that causes this to be so.

The Project has never conceived of its job in such a fashion. The task to which it has addressed itself is indicated by the fact that the mathematical curriculum in our elementary and secondary schools has appeared to many knowledgeable observers to be a poor one; the experiences that both teachers and children have had with mathematics have seemed unsatisfactory; and, what is by far the most alarming aspect of the matter, efforts to improve these experiences have seemed to bear little fruit.

This last difficulty is not easily identified, but in the view which the Project has come to adopt, the problem is partly epistemological, and involves two main components: first, a tremendous gap in sophistication, focus, or point of view between the language and interests of teachers, as contrasted with the language and interests of "educational research" (or, indeed, of university professors in general, regardless of department); and, second, an equally large gap between the kind of knowledge which teachers need, as contrasted with that which the conventional wisdom of education seeks to accumulate. Although Michael Polanyi does not specifically have education in mind, his description of the distinction is one of the most articulate in print.[54]

The conventional wisdom of educational research takes the task of improving our schools and assigns it to an accumulating body of abstract generalizations. Yet even when one looks within this very body of generalizations it becomes clear that institutions and people are influenced mainly by institutions and people. Teachers do not teach as they were told to teach; rather, they imitate their own teachers, and teach as they themselves were taught. In a contest between "doing what I say" versus "doing what I do," rationality usually comes in a poor third, if it enters the race at all.

The present Project has sought to produce certain actual changes in schools. Seen from the point of view of the conventional wisdom of education, this is epistemological heresy. The Project has not sought to prove that these changes were desirable, any more than it would have sought to prove that the introduction of the dominant major ninth into classical harmony, or the introduction of the clarinet into the classical orchestra, was desirable. These are things that you do, and then allow people to view them, and to build upon them, and to form their own judgements.

This is not a denial of responsibility, nor a defense of secrecy. It is, instead, a claim that values are, indeed, a matter of values; that people disagree about values; and that simple empirical criteria cannot conceivably settle such disagreements.

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[54] Polanyi (77); cf. also Boulle (4).

In any case where "goals" can be agreed upon, and where a range of acceptable methodology can be agreed upon, one can easily seek to achieve (or, if they are measurable, to maximize) these goals by selecting within the range of acceptable methodology. Such cases are extremely rare, and they do not embrace any large part of education.

Consequently, what is presented in this "Results" section of the present report is not the results, but rather a very partial description of some of the results. From the Project's point of view -- and, therefore, within the present report -- the most valuable sections of a reported innovation are those which are entitled "Purposes" and "Methods." The "results" themselves have occurred to real teachers and real children in real classrooms.

Those who have worked on this Project want their efforts to be understood. They even wish their errors to be recognized and corrected. In choosing to speak the language of the classroom teacher, rather than that of the "educational researcher," they have been seeking what they believed to be the most direct and responsible way to make certain changes in schools. In choosing media to present these changes, they have selected primarily these two:

i) films showing actual "process" lessons;

ii) workshops involving face-to-face confrontations where teachers (or others) could experience for themselves some of the experiences in question.

(A third medium of reporting has been the written a posteriori "lesson plan.")

This raises one of the hardest epistemological questions with which we are faced today: is this the proper kind of "knowledge" to acquire or to transmit?

That such films exist is beyond dispute; so is the fact that such workshops have been held, and have by now been attended by thousands of teachers in greater St. Louis, in Fairfield County, Connecticut, in San Diego County, in Los Angeles, in Chicago, in New York City, on Long Island, in Philadelphia, in Washington, D. C., in Dade County, Florida, in Corpus

Christi, Texas, and elsewhere.

How, then, shall we describe the "results"?

In the first place, the serious student is asked to view as many films as possible, and if possible to attend one or more of the workshops. In this way he will come to know something of the results themselves, rather than to know merely a description of the results.[55]

It is also impossible to comprehend the "results" without some regard for the purposes. The "purposes" of this Project -- as with several others -- are frequently misunderstood. One can regard the school curriculum in mathematics and science as consisting of the earlier steps in a ladder intended to culminate in the production of professional scientists and mathematicians, and to keep one's view firmly fixed on this goal. This is not the focus of the Project's attention.

Peter Schrag has stated polar objectives for today's education, in his remark that:

> It was comforting to believe that our children would understand the virtues of our affluent, technological consumer society and would rush to join it.
>
> Many, of course, still do, and many are, indeed, indistinguishable from their predecessors, and

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[55] In an age as sophisticated as ours sometimes seems to be, one is continually astonished that our most serious problems are of a notably primitive sort. One is surely almost embarrassed to stress the error of confusing reality with descriptions of reality, yet precisely this confusion is one of the errors presently impeding human progress on many different fronts. The point is significant because all descriptions share the common feature that they are different from the reality which they attempt to describe. In this sense, every description is wrong. If such a thing as a "correct" description existed, it might indeed possess the magical properties that are improperly ascribed to real (i.e., wrong) descriptions.

> yet this is still a new September. It is a September when the big facts are not school facts and when the most valuable social commodity is not money or labor or land but skill, knowledge, and training. As John Kenneth Galbraith suggests in *The New Industrial State*, it puts the educational institutions, and especially the universities, into key positions of power and gives them bargaining leverage that they have never had before. It gives education a new degree of freedom to criticize and to dissent, and *it confronts every school and every college with a choice between training useful cadres for the system and developing free human beings who can pit their humanity against the subtle but awesome pressure of a society ever more prone to subtly manipulating its citizens.* [Italics added by RBD]
>
> This September is different because more than ever before the children who come into the classroom have lived and understand things that we only know secondhand: Our world was made of the dreams of the depression, furnished with little houses and picket fences and a car in every garage; theirs is made from the nightmare and hollowness of its reality and of the dreams of peace. We looked ahead to prosperity and victory; they can look ahead to -- what? It is different because what education must give them is not facts but experience, not programs but engagement.[56]

Obviously, any effective combination of curriculum and pedagogy can serve alternative masters, at least with a few minor adjustments here and there. However, the goal of the present effort was Schrag's second alternative: to "develop free human beings who can pit their humanity" against all the diverse pressures and problems of their environment.

An earlier age would have given this assignment to theology. The Renaissance might have assigned such a task to the rediscovery and reinterpretation of the ancient world of the Greeks and the Romans. Today, for a variety of reasons, science and mathematics lie closer to the question of who man can become, and what society ought to do.

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[56] From: Peter Schrag, "September Song," *Saturday Review* (September 16, 1967), p.87. Reprinted by permission of Saturday Review and Peter Schrag.

The traditional arithmetic curriculum could be attacked as, among other things, a menace to mental health; an abstract ritual, devoid of meaning, to be learned precisely with no allowance for error or originality, it has been denounced by children and adults alike since the time of Macaulay (cf., e.g., Winston Churchill's remarks on the subject). To this mainly negative ritual there was added the further insult of some exceedingly middle-class (and quite wrong) discussions about buying annuities, investing money, and managing checking accounts.

That arithmetic should be cast in the role of compulsive villain is in some ways particularly ironic, since it ranks high among the subjects which can be enjoyable and liberating to children. Mathematics, science, and art are especially suitable to provide opportunities for growth, for a pleasant partnership of teacher and student, and for heightened self-understanding. A large part of our accumulated culture is embodied in mathematics, science, and art; these subjects are among the least controversial in the curriculum (by contrast, history, economics, and political science combine the liabilities of controversy with a propensity to degenerate into brain-washing, and in response to such threats they usually become unappetizingly bland); mathematics and science allow the child room to explore -- one of the child's favorite activities whenever it is permitted -- and even have the virtue of being largely self-checking. A mathematical guess can be tried out, and one knows quickly whether or not it has worked successfully.

This last point -- which the Project refers to under the heading of autonomous decision procedures -- is so important that four examples will be given.

Autonomous decision procedures

Example 1: Counting. As soon as a child can count discrete objects (such as pebbles or bottle caps) reliably, he can verify for himself the truth or falsity of a proposition such as

$$3 + 4 = 7,$$

by counting out 3 pebbles, then counting out 4, then pushing them together and counting the result. He is free from the authority of the teacher in a way he cannot be when he is studying poetry or history.

Example 2. Given the open sentence

$$(2 \times \square) + 3 = 8,$$

a child who can do a little arithmetic can guess an answer -- say 2 -- and try it out for himself to see if it works:

$$\begin{aligned} (2 \times \boxed{2}) + 3 &= 8 \\ 4 \quad + 3 &= 8 \\ 7 &= 8 \qquad \text{False.} \end{aligned}$$

The child sees for himself that $2 \longrightarrow \square$ will not produce a true statement.

Example 3. Wallace Feurzeig of the Cambridge firm of Bolt, Beranek, and Newman has reported on an experiment in letting twelfth graders program digital computers. Nearly all computer programing requires a high degree of precision -- the computer makes no allowances for good intentions. Every symbol must be *exactly* correct. In this regard, computer programing seems to resemble a compulsive approach to traditional arithmetic, or to the study of traditional grammar, punctuation, and spelling. It is, however, entirely different. In these traditional subjects one found a contest between teacher and student, in which the teacher seemed cast in the role of inflexible tyrant. Student resentment against capricious *human* tyranny is not so readily mobilized by a mere machine, and the demand for precision is *acceptable* in the case of a machine, but *unacceptable* from a human being. (Madison Project experience in allowing fifth, sixth, and seventh graders to program a digital computer exactly parallels the Bolt, Beranek, and Newman experience in this regard.)

Example 4. This is, in fact, a reasonably correct anecdotal account of one of the Project's earliest experiences. The students were very low achieving seventh graders whose arithmetical skills were quite limited, and who tested low on group I.Q. tests (around I.Q.'s of about 80). These students had been taught how to plot points in Cartesian coordinates, and, working as individuals or in small groups of two or three, they had made tables and graphs (incomplete, of course) for the truth set of open sentences such as

$$(2 \times \square) + 3 = \triangle.$$

Several boys, discovering the "slope" pattern for integer replacements of the variable $\square$, constructed the graph *geometrically* by extrapolating via this pattern, got some point involving relatively large numbers (such as $10 \rightarrow \square$, $23 \rightarrow \triangle$), and then substituted these numbers into the equation to see if the resulting statement was true. Their pleasure at the observation that this always worked was unmistakable, and Project teachers conjectured that this is one of the first things these children had learned in school that checked against their own personal experience. How much further joy came from the fact that they had "discovered this for themselves" is hard to assess, but the "autonomy" and "origin-pawn" experiments of Richard deCharms suggest that this factor is probably not negligible. Some children took a large supply of graph paper home, and spent hours repeating this experience with different equations which they made up themselves.

In considering the following discussion of "Results," the reader is then asked to bear in mind that the goal was to change the school which frustrates teacher, child, and parent alike into an institution where adults and children could work side by side, exploring the legacy of our culture, exploring the possible wise uses of human intelligence, and pursuing continuing growth toward autonomous maturity. The "weaknesses" of the school which the Project has sought to remedy are not primarily related to the Russian launching of Sputnik ahead of the United States; rather, they are the kind of weaknesses identified by Paul Goodman, by Peter Schrag, by Pierre Boulle, by David Hawkins, by Haim Ginott, by Jules Henry, by Carl Rogers, by Richard deCharms, by John Holt, by Edgar Z. Friedenberg, and even by John Dewey. They have been chronicled by Lawrence Cremin,[57] dramatized by Bel Kaufman's *Up the Down Staircase*, and recorded with considerable precision by Louis Smith and William Geoffrey.[58]

The purpose of the Project has been to attempt to change this disaster into something gratifying.

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[57] Cf. Cremin (17).

[58] Cf. Smith and Geoffrey (97); cf. also Hawkins (47); and also Featherstone (35), and Featherstone (36). This second Featherstone article is probably the best statement of what the Madison Project is seeking to achieve that presently exists in print, although Featherstone is of course discussing schools in England, and is not intentionally discussing the Madison Project at all.

Having disposed of the preceding necessary words of warning, we now discuss the actual "Results":

A. The "Curriculum" Aspect

From the point of view of the classroom (in grades K-9, primarily), the Project has produced a sequence of curricula, most of which are intended to be supplementary to the school's regular curriculum in science and mathematics.

These curricula combine "content" and "pedagogy"-- they are perhaps not so much a "curriculum" in the traditional sense as they are a sequence of quite fully formed experiences. The role of the teacher, and the roles of various students, are spelled out quite explicitly, and recorded on 16 mm. film. This may appear to be very stereotyped and anti-creative. In fact it is not. Beethoven clearly "imitated" both Mozart and Haydn, Bach "imitated" numerous of his predecessors, Picasso "imitated" Degas, and neither Beethoven, nor Bach, nor Picasso seem to have been rendered thereby less creative. Imitation in this form is perhaps nothing more than one stage in effective communication. The teacher is not left to *imagine* what is meant by a lesson -- she can see *exactly* what is meant, at least as accurately as the best contemporary television technology and artistry can record an incident of human behavior. (And in the case of face-to-face workshops the teacher will either actually experience the lesson as a student, or will actually practice teaching it, or both.)

A list of various curricula developed by the Project over the years is presented in Appendix G. The most important of these curricula are the following five:[59]

i) *The basic supplementary "unified" curriculum for grades two through eight.* This "curriculum" (which we shall designate "curriculum α") consists of a sequence of quite specific lessons, spelled out in considerable detail and recorded on film. Its purpose is to provide a foundation for combining arithmetic, algebra, geometry, and science. It has been tried with a wide variety of children, including many urban children presumed to be underprivileged. (The performance of the underprivileged children has not been very different from that of privileged children, so far as this curriculum is concerned.) This curriculum seems to be reasonably stable for many different kinds of children, and makes modest demands upon the teacher (but most teachers could not teach this curriculum satisfactorily without a moderate amount of special training).

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[59] We are again concerned with "descriptions" vs. reality. We know of no way to determine whether two "different" curricula are "really the same." Probably no teacher ever teaches "the same curriculum" on two different occasions, and two different teachers never teach "the same curriculum." How broadly or how sharply defined should "curriculum" be?

A more detailed presentation of this curriculum will be given below, after an introductory comparison of all five curricula.

ii) The "assembled" curriculum for grades two through eight. Experience with urban children using "curriculum α" led to the conclusion that it was probably deficient in the following respects:

a) there was probably too much teacher-domination of the learning environment;

b) there was probably too much work with the entire class, and not enough "small group" work;

c) there was insufficient diversity in the kinds of activities available for the children;

d) there was insufficient relation to arithmetic;

e) there was insufficient use of physical materials and of a multi-sensory multi-media approach;

f) "curriculum α" was probably too easily assimilated into usual curriculum and pedagogy procedures, in the phenomenon that is sometimes referred to as "nullification by partial assimilation," without producing the changes that it was intended to produce.

At the same time that these deficiencies were being noted in "curriculum α" in the United States, almost identical deficiencies were being noted (or had been noted slightly earlier) in England, and some English educators had found important innovations that went far toward overcoming these deficiencies. This was conspicuous in work done by Leonard Sealey, Edith Biggs, Geoffrey Matthews, and Z. P. Dienes, as well as by other members of an unusual British organization known as the Association of Teachers of Mathematics.

The Madison Project, having undertaken (under National Science Foundation support) to assist the cities of Chicago, New York, Philadelphia, Los Angeles, and the county of San Diego with special teacher workshops in curriculum and instruction in elementary school mathematics, assembled a considerably broader curriculum based upon:

a) "curriculum α," as discussed above;

b) the English work of Sealey, Biggs, Matthews, Dienes, the Nuffield Mathematics Project, and various members of the A.T.M.;

c) the work of the Elementary School Science project of E.D.C., in Newton, Massachusetts (originally under the direction of David Hawkins, and itself heavily influenced by Leonard Sealey and the British group);

d) additional classroom lessons, using physical materials, newly devised by Madison Project staff, or adapted by Project staff from other earlier materials (such as the "geoboard" used by Caleb Gattegno and George Polya).

During August, 1967 (for the purposes of the workshop in New York City), another ingredient was added, namely, the elegant "outdoor mathematics" developed by Professor Lauren Woodby of Michigan State University (at East Lansing).

The resulting curriculum, related more closely both to arithmetic and to science, using small-group classroom organization, and emphasizing multi-sensory experiences with physical materials, is clearly much richer in diversity of classroom experiences, and seems to have greater appeal for most children. It represents an unmistakable departure from traditional school practice in the United States. We shall refer to it as "curriculum β." Very little, if any, of "curriculum α" need be sacrificed in order to use "curriculum β."[60]

iii) A simplified curriculum for primary grades (specifically, for nursery school, kindergarten, and grades one and two). All Madison Project materials are intended to be ungraded (which incidentally retards the speed with which they can be implemented in most schools), so all grade-level designations should be considered only approximate guidelines, at best.

Experience in operating workshops for teacher education, especially in the major urban areas, indicated that one basic need was not for a "fancier" program for young children, but rather for one that was simpler and more natural. Quite young children easily verbalize number ideas (such as the spontaneous remark from a boy, aged three years, eight months, "There are three orange trees" -- which was correct), and the Project has been trying to build on this -- including such topics as learning about United States currency ("When I get four quarters, I'll take them to a bank and get a real dollar" -- a spontaneous remark from a girl, aged five years, three months, who in fact had one quarter at the time) and trying to build upon others of young children's natural mathematical remarks concerning the world around them. Here, too, the British have valuable contributions to offer which are gratefully received into the Madison Project curriculum.[61]

We shall label this "curriculum γ." It is an anomaly among "new mathematics" programs in the United States in that it avoids formalism, avoids the specific notion of "set" (in the mathematical sense), is based primarily upon the act of counting, seeks to utilize the child's natural modes of learning (and especially to utilize the child's most natural methods for naming or indicating numbers, which often consists of holding up the correct number of fingers), and accepts most mathematics of this age level on what Polanyi would presumably classify as an "unexamined basis," selecting mathematical ideas (or language) for specific examination only in relatively infrequent cases, and only where the "meta" examination might be presumed to confer some specific (and relatively immediate) benefit.[62]

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[60] Independently, and more-or-less simultaneously, the Cambridge Conference on School Mathematics, meeting at Pine Manor Junior College in August and September of 1967, developed the broad outline (and some of the detail) for a combined-math-and-science curriculum for elementary schools which parallels surprisingly closely "curriculum β." "Curriculum β" is, of course, for the most part worked out in relatively complete detail, and is somewhat more conservative than the CCSM proposals, because of its direct relation to more-or-less immediate implementation in thousands of urban classrooms in the five participating cities (New York City, Philadelphia, Los Angeles, Chicago, and metropolitan San Diego).

[61] Cf. especially Featherstone (36).

"Curriculum γ" is a particularly old-fashioned-looking piece of "new mathematics." One of the tasks for the future, quite clearly, is to relate "curriculum γ" in a natural way to appropriate portions of "curriculum α" or "curriculum β."

The general flavor of "curriculum γ" can, at least to a limited extent, be inferred from the a posteriori "lesson plans" written by Beryl S. Cochran, and included in this report as Appendix E.

iv) The Ninth-Grade Course ("curriculum δ"). The work with curriculum α, with curriculum β, and with the fifth curriculum discussed below ("curriculum ϵ"), led to the question of "where children would go after completing any of these programs." Probably only "curriculum ϵ" raises this question seriously, but it seemed advisable to explore possible answers. The ninth-grade program, intended for the population of college capable students, was developed in order to be able to exhibit at least one possible answer.

There was a second reason for developing the ninth grade course. The Project's attempts to develop appropriate courses for grades seven and eight have consistently failed, except under special circumstances. This has led to the conjecture that possibly these years in a child's life either call for no study of mathematics, or for a radically different form of mathematical study, or (as seems likely) for a radically different type of school experience (quite possibly so different that most people today would not consider it "school" at all).

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[62] Cf. especially Hawkins (44).

The obvious response to these difficulties in grades seven and eight (coming on the heels of consistent successes in grades four through six) was to try to determine the dimensions of the problem, by trying to identify some point "on the other side of the age gap." Hence the attempt to teach a ninth grade course.

The ninth grade course was taught for two consecutive years to ninth grade students at Nerinx High School, in Webster Groves, Missouri. It has been extensively recorded on film, and is described in detail in Davis (22).

It turns out that, for college-preparatory students of the type who attend Nerinx High School, grade nine is, indeed, "on the other side of the gap," and the difficulties that plague grades seven and eight do not appear in teaching the Nerinx ninth graders.

Features of the ninth grade course include:

a) It is a complete course, and not "supplementary" as most other Project programs are.

b) It was based upon an attempted identification of the weak spots in traditional (and most "modern") curricula, including:

> the absence of logic and an axiomatic approach to algebra
>
> an inadequate utilization of analytic geometry and the algebra of matrices
>
> no adequate treatment for the concept of limit of a sequence
>
> no relation to experimental science
>
> no opportunity for the students to develop mathematical systems for themselves
>
> no systematic way of proceeding from initial intuitive ideas to increasingly explicit and formal versions (again, with the students taking the lead in this development)

insufficient consideration of the diversity of possible mathematical systems.

c) Because of b), the course does use logic and an axiomatic approach to algebra, it does use analytic geometry and the algebra of matrices, it includes the study of limits for bounded monotonic sequences (which suffice for the work at hand), it involves some actual physical experiments, it allows students to develop various mathematical systems themselves, it moves carefully from initial intuitive ideas to subsequent formalizations, and it shows clearly the diversity of possible mathematical systems.

d) Some emphasis should be placed on a point already mentioned: the course includes a few actual physical experiments, to be performed by the students. This has implications for school architecture (at least to the extent of "borrowing" science classrooms for these particular lessons).

e) The course combines "small group work" as a principle of classroom social organization, to be used about two-thirds of the time, with total class "large group" discussion, to be used about one-third of the time.

f) In its present form, the course can be taught only by specially-trained teachers.

g) The course does *not* presuppose any "modern mathematics" programs in grades K-8, but students who *have* had previous Madison Project courses consistently outperform those who have not. This is presumably a quite unsurprising consequence of continuity of viewpoint, notations, etc.

h) Whereas curricula β and γ, especially, are intended for *all* children, this ninth grade program is intended for all college-capable children.

v) *The "sophisticated" curriculum for grades three through eight* ("curriculum ϵ"). Whereas each of the preceding curricula are in some sense "successful" and (within reason) "reliable," this present fifth curriculum is something of an oddity, apparently not really a viable curriculum, yet tantalizingly provocative in those cases where it has succeeded.

All of the first four curricula listed here are, to a greater or lesser degree, "practical solutions" to present school needs. This fifth curriculum is not. It is listed as important more because of the theoretical questions which it appears to raise.

From autumn, 1959, until June, 1964, the Project conducted a very sophisticated mathematics program for the upper third of the students in Weston, Connecticut (which had at that time a three-track program), for grades three through eight. This program is extensively recorded on film and on audio-tape (it is the most fully documented of all of the Project's programs; for several years every single lesson was recorded on audio-tape, and these tapes have not been erased; a few were lost in a fire when one of the Weston school buildings burned down).

This program in Weston appeared to all observers and by all criteria used to be superbly successful;[63] when, however, it was subjected to viability testing by attempting to have the same teachers replicate it in other school systems, the program was unsuccessful and failed to hold the students' interest (in Weston, the same students were followed for five consecutive years, and their performances are recorded on tape and film over this period). When it was further subjected to feasibility testing with different teachers, it also failed (although on achievement tests the students outperformed carefully matched students in more conservative curricula). A few students became captivated and highly involved, but most were judged by teachers and observers to be relatively apathetic (though surely no more so than in traditional classes).

Why this program worked so well in Weston, Connecticut, and has never worked equally well elsewhere is entirely a matter for conjecture.

B. A More Detailed Look at the Five Basic Curricula

"Curriculum α." Purpose: A supplementary program intended to enable schools to relate the study of arithmetic, algebra, geometry, and science, thereby creating a single unified math-science curriculum

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[63] Cf., e.g., the achievement test study by J. Robert Cleary of Educational Testing Service, Appendix A, pp. A-6 through A-25; more striking evidence is available in the films showing the actual classroom lessons.

(non-graded, but useful in grades two through eight).

Concepts and skills involved: plotting points in Cartesian coordinates, arithmetic of signed numbers (i.e., integers: positive, negative, and zero), variables, functions, methods of representing functions (including algebraic formulas, tables, and graphs), measurement via inequalities, measurement via numerical answers with uncertainty, open sentences and truth sets, angles (regarded as rotations), linear measurement, area, and volume.

Examples of specific classroom experiences:

1. *Introduction of unexamined use of language in the case of the concept of "variable."* We have considered earlier (cf. "Remark 2" on p. 13) Michael Polanyi's distinction between "examined" vs. "unexamined" use of language -- which means especially the "unexamined" *introduction* of new language vs. the "examined" introduction. We have also considered the problem of selecting among the alternative meanings of "variable" (cf. pp. 50-51).

Once the choice of meaning for "variable" has been settled -- and for present purposes this is the meaning used in contemporary mathematical logic (and, say, in the UICSM program): "a variable is a placeholder for the name of a number (or other mathematical entity)," we must choose either an "examined" or an "unexamined" introduction in the sense of Polanyi. The Project, in its work with younger children, has consistently found an "unexamined" introduction to be more effective.[64]

This clarifies the task considerably: we want to get children -- say second-graders (i.e., about eight year olds) -- *working* with variables and *discussing what they are doing without* too much self-consciousness.

One simple solution to our problem -- once it has been posed in this form -- is to write

$$3 + \square = 5 ,$$

and to ask the child:

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[64] As noted on p. 14, this is contrary to the usual practice of most schools at the present time.

"What can I write in the box to make a true statement?"

"What can I write in the box to make a false statement?"

This precise lesson, with children of various grade levels, can be observed on several Madison Project films, including the film entitled First Lesson.

2. Practice with addition and multiplication, practice with the concept of "variable," and an opportunity to discover some patterns and to make use of them. The Project has argued that "rote drill" is always a confession of failure on the teacher's part, tantamount to saying: "I can't find any sensible reason for you to do this, so do it just because I told you to."

Practice, then, should be embedded into various interesting tasks[65] wherever possible, and these tasks should provide for considerable practice in the processes of mathematics (such as "discovery of pattern," "clarification of meaning," etc.)

For practicing addition and multiplication, as well as the use of variables, we have a natural task (which Project teachers learned from observing children, and did not devise a priori): namely, seeking truth sets for quadratic equations.

One can start with quadratic equations so simple that children, using trial and error, will surely quickly hit upon the correct answer; e.g.

$$(\square \times \square) - (5 \times \square) + 6 = 0 ,$$

and one can work up to such difficult problems as

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[65] In fact, virtually all of school can be "embedded in intrinsically rewarding tasks" if we go about it correctly. One astute observer said of the excellent Hilltop School in Ladue, Missouri: "This isn't a school; this is a place where you go to have fun and to learn things!"

$$(\square \times \square) - (20 \times \square) + 96 = 0 ,$$

which surely provides considerable practice in addition and multiplication for most students.[66]

3. *The explicit or "examined" use of language.* If ordinary school practice often "tells" children things that they do not need to be told, it also often fails to emphasize matters which *do* need to be told explicitly. One such case is the "rule for substitution" in using variables.[67] The Project states this explicitly, since it is essentially an arbitrary (though highly convenient) *agreement*, and by no means a "law of nature" which might be discovered.

Besides stating explicitly the "rule for substituting," and introducing the explicit notation

$$\text{U.V.: } 5 \longrightarrow \square,$$

the Project (inevitably) later gives explicit recognition to the *replacement set* for the *variable*. This explicit discussion would come around grade four for verbally-gifted children, whereas the actual use of variables (in "unexamined" form) would begin at least as early as grade two, even for quite non-gifted children. (Cf. the film *First Lesson*, and the film entitled *A More Formal Approach to Variables*.)

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[66] Obviously, it is *not* intended to suggest that a sequence on quadratic equations would provide the *sole* practice in adding and multiplying; quite the contrary -- a wide diversity would be used.

[67] Unfortunately, different authors reverse the meanings of the words "substitution" and "replacement," the common practice of engineers being, for example, at variance with the usual practice of mathematical logicians. The Madison Project has therefore resorted to a new notation: "U.V.," meaning "use of a variable." This avoids any doubt as to which process is meant.

4. Introduction of signed numbers: the "pebbles-in-the-bag" model. A method of making curriculum choices was discussed earlier (cf. p. 50), under the name of the "topic-extension" approach. The present example illustrates this, and also illustrates two of the Project's basic approaches to designing classroom experiences, which will be discussed presently.[68]

There are many different theoretical interpretations of "signed numbers" (i.e., integers: positive, negative, and zero; or rational numbers, of each type), and it would not be useful here to consider them in discussing the present example (cf. pp. 52-56).

What is relevant is that, as discussed on pages 50 through 56, the "topic-extension" approach to designing curricula virtually forces us to provide an answer to the question

$$4 - 6 = ? \quad .$$

(This type of question, for example, will arise when we consider

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

if we use 4 as a replacement for the variable $\square$.)

The further design of a suitable classroom experience (within customary Madison Project procedures) is guided by certain general precepts which are not scientific generalizations but rather habitual ways of conceptualizing teacher-learning experiences. Two such precepts, which play a major role in the present example, are these:

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[68] Again, there is a serious epistemological problem: over the past ten years the Madison Project curriculum designers and specialist teachers have developed a rather useful "in-house" vocabulary for discussing the process of designing actual classroom "happenings." How useful is it to record this "theory" in writing, in order to make it available to others? (Actually, it is probably not properly called a "theory," but rather a combination of conceptualizations and "practitioner's maxims" in the sense of Polanyi.) Effective work does not just happen; it results precisely from such a combination of conceptualizations and practitioner's maxims. Could Beethoven have described in words how he designed a musical composition? Would it be useful to others if he had?

The "paradigm" model of knowledge. It is quite clear that the actual data-processing which human beings perform is exceedingly complex, yet for purposes of designing classroom experiences one needs some conceptualization of this, preferably a quite simple one. The Project ordinarily uses the "paradigm" conceptualization: "knowledge" which a person possesses is, in this conceptualization, regarded as a specific past experience stored more-or-less in toto in memory, much as a short episode might be recorded on motion picture film, but with the added capability of being somewhat modified, so that variations on the original episode can be (to continue the metaphor) projected on the screen, and not merely identical repetitions of the original episode. This "original episode" is what is here being called the "paradigm," whence the name of this conceptualization.

The "do ... then discuss" strategy for teaching. A corollary of the "paradigm" conceptualization of knowledge is the "do ... then discuss" strategy for teaching. This is in some ways akin to taking a class of children to the zoo, then asking them to talk about what they saw and did.

This strategy calls for the teacher to do something together with the children, or to have the children do something themselves, after which children and teacher discuss what was done. (This is a quite common strategy in many schools in dealing with younger children, but is not used much after, say, about the third or fourth grade; the Project uses it at all grade levels.)

Applying these two notions to the task at hand, we want to "do something" with the children which will give them an "episode" or "experience" with (say)

$$4 - 6$$

which can thereafter be discussed, modified as necessary, and so forth. Within Project language, the teacher is developing in the child's mind some appropriate mental imagery (the basic "paradigm" or "model" or "experience" to give meaning to "4 − 6").

Clearly, "4 − 6" is to be interpreted within the reference point system (cf. pp. 55–56) rather than in the "counting" or "how many" system (cf. p. 55).

The curriculum design problem is now fairly clear: find a concrete experience within a "reference point" setting that can serve as a general model for problems of the type

"4 − 6 = ?" .

The Project's actual response was devised in working in equatorial Africa, where temperatures below zero were not readily available for building experiences, nor could one use ammeters nor double-entry bookkeeping, deficit spending, and credit. Instead, the experience chosen goes as follows:

A bag is partly filled with pebbles (containing enough pebbles, in fact, so that it will probably not be emptied by the ensuing transactions). There is a separate pile of pebbles available on a table nearby.

We shall focus attention on the question of how many pebbles there are in the bag -- but this must not be done as a counting problem, but rather as a reference point problem. Hence we shall establish a reference point by some suitably dramatic occurrence (as years in the Christian system are counted from the birth of Christ, unfortunately with the omission of any year "zero"). The one used is to have some child (John, say) shout "Go!" This "starts the game" -- i.e., it establishes our reference point. We shall not ask: "how many pebbles are in the bag" as a counting problem, referring to the task of counting all of the pebbles in the bag. Instead, we shall discuss "how many pebbles in the bag" as a reference point question -- are there more pebbles in the bag than there were when John said "Go!," or are there less, or are there the same number? [Although this would not be discussed explicitly with the children, we are making an "unsymmetric" use of the symbol "=", in which numerals on the left refer to counting operations, while numerals on the right refer to the condition of the bag, described in reference point terms. Notice that what is done is simple, but the task of describing what is done in relation to sophisticated contemporary "meta" discussions of mathematics is somewhat complex.]

A possible "happening" might proceed as follows:

John: Go!

Teacher: How many pebbles shall we put in the bag?

Mary: Five

Teacher writes: 5 [while a child actually puts five pebbles into the bag][69]

Teacher: How many pebbles shall we take out of the bag?

Nora: Seven

Teacher writes: $5 - 7$ [while a child again actually physically removes seven pebbles from the bag][70]

Teacher: Are there more pebbles in the bag now than there were when John said "Go," or are there less?

Children: Less

Teacher: How many less?

Children: Two less

Teacher writes: $5 - 7 = {}^{-}2$ [and relates the symbol and name "negative two" to the condition of having "two less pebbles in the bag than when John said 'Go' "].

This can be viewed in many Project films, perhaps especially the film A Lesson With Second Graders. (Notice that the language and symbols are introduced largely in an unexamined way, with as little explicit discussion as possible.)

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[69] For the sake of "meta" discussions, notice that this was a bona fide "counting" transaction.

[70] Again, a bona fide "counting" operation -- the child counts out seven pebbles.

"Curriculum β." (The "assembled" curriculum for grades two through eight, drawing heavily upon E.S.S., the Nuffield Mathematics Project, etc.)

Purpose: As with "curriculum α," this curriculum is designed to provide a foundation for relating closely together the areas of arithmetic, algebra, geometry, and science. "Curriculum β" however goes much further. The classroom social organization is changed to emphasize small group work, individualized instruction, the use of physical materials, the use of "mathematics laboratories," etc. These departures from "curriculum α" have been discussed earlier in this report.

"Curriculum β" carries the "unification" theme further, including carpentry, social studies, art, communication skills, etc., along with mathematics and science. It is intended to reach a greater diversity of children, and appears to be able to do so.

Actual classroom lessons of Madison Project "curriculum β" can be seen on the films Geometry Via Concrete Objects, Gluing and Stamping, Using Geoboards with Second Graders, An Introduction to Geometry Via Nailboards, A Sixth-Grade Lesson on Place-Value Numerals, The Concepts of Volume and Area, The Classroom Divided Into Small Groups: Counting, Volume, and Rational Approximations, Small-Group Instruction: Signed Numbers, Rational Approximations, and Motion Geometry, and various others.

Three films not made by the Madison Project relate to this general theme, and possibly express the main ideas more clearly, namely: I Do ... And I Understand, available from Mr. S. Titheradge, Manager, New Print Department, Sound Services, Ltd., Wilton Crescent, Merton Park, London, S.W. 19, England; Maths Alive, available from the Foundation Library, Brooklands House, Weybridge, Surrey, England; and Classrooms in Transition, available from Mary Lela Sherburne, Education Development Center, Inc., 55 Chapel Street, Newton, Massachusetts 02158.

"Curriculum β" is sufficiently rich in content diversity that different portions of it may be made to serve quite different purposes. It includes "curriculum α" as one of its parts (but due to individualization of instruction, not all children would necessarily meet all parts of "curriculum α"). In addition there are additional units developed by the Madison Project, especially in relation to arithmetic (e.g., extracting arithmetic problems from the morning newspaper), and to geometry via geoboards. It includes also units developed by other individuals and groups, sometimes modified and sometimes used without modification:

many units borrowed from the Nuffield Mathematics Project

Marion Walter's "Mirror Cards" (E.S.S.)

Marion Walter's "informal geometry"

attribute blocks (E.S.S.)

Cuisenaire rods

various units developed by Leonard Sealey (e.g., on *pronouncing* names for numbers, and *writing* names for numbers by using Dienes' MAB blocks) and by Edith Biggs (e.g., on classification of geometric shapes of naturally-occurring objects, such as an assortment of cardboard boxes)

the use of a wide variety of desk calculators, such as the Lagomarsina, Monroe calculators, ten-key calculators, full-keyboard calculators, double-keyboard calculators (of the type commonly used by statisticians), calculators that print on paper strips, inexpensive plastic calculators costing a dollar or so, manual machines, electric machines, etc.

Lauren Woodby's "outdoor mathematics," emphasizing *measurement*, *ratio*, and *proportion*

other uses of Dienes' MAB blocks

place-value numerals using beans, tongue depressors, etc. (developed by Beryl Cochran)

the "sine-generating machine" developed by the Cambridge Conference on School Mathematics during the summer of 1967

the simple rough study of periodic functions, such as temperature at various hours of the day (suggested by Professor Andrew Gleason of Harvard University)

other units involving periodic and sinusoidal functions (some developed by Donald Cohen, and some adapted from E.S.S.).

It should be emphasized that "curriculum β" can be used so as to be *easier* than "curriculum α," or so as to be *more sophisticated and more profound*. The desired outcome is achieved by individualized programming for different students or different classes.

"Curriculum γ." (a simplified curriculum for primary grades). Purpose: To build as naturally as possible on a young child's propensity for seeing mathematical aspects of his environment, provided he is given reasonable encouragement.

The best description of "curriculum γ" at present is Beryl Cochran's report which appears as Appendix E to this report.

Films showing lessons in "curriculum γ" include: the series of five films entitled *Teaching Big Ideas in Mathematics to First Grade Pupils*, *Addition and Multiplication Using Plastic Washers*, *Multiplication Using Dots*, *Experience With Fractions -- Lesson I*, *Experience With Fractions -- Lesson II*, and others.

The non-Madison Project film *Maths Alive*, mentioned earlier, is also relevant here.

"Curriculum δ." Cf. the ninth grade films listed in Appendix B, and the ninth grade report mentioned earlier.

"Curriculum ϵ" (the "sophisticated" curriculum for grades three through eight). Concepts include (in addition to all those of "curriculum α"): axiomatic algebra, implication, contradiction, uniqueness, isomorphism, the algebra of 2-by-2 matrices, truth tables, inference schemes, frequency distributions in statistics, graphical integration and graphical differentiation, derivation of the quadratic formula, vector kinematics, velocity, acceleration, and the ϵ,N-definition of limit of an infinite sequence (together with formal proofs, using this definition, that the limit of a sum is the sum of the limits, etc.).

As discussed earlier, actual classroom lessons from "curriculum ϵ" are more fully recorded on audio-tape, video-tape, and film than is the case for any of the other curricula discussed here. Films include: *Clues*, *Matrices*, *Solving Equations with Matrices*, *Accumulating a List of Identities*, *Introduction to Derivations*, *Second Lesson*, *The Study of Functions -- Linear, Quadratic, and Exponential*, *Small-Group Instruction: Signed Numbers, Rational Approximations, and Motion Geometry*, *Small-Group Instruction: Committee Report on Motion Geometry*, *Average and Variance*, *Programming the IBM 1620, Using GOTRAN*, *Derivation of the Quadratic Formula -- First Beginnings*, *Derivation of the Quadratic Formula -- Final Summary*, *Complex Numbers via Matrices*, *Jeff's Experiment*, *Graphing an Ellipse*, *Limits (First Version)*, *Limits (Second Version)*, and others.

C. Classroom Atmosphere

The emotional climate, the means of communication (both verbal and non-verbal), the respective roles of teachers and of students are what they are. It was largely because of the unlikelihood of an adequate rhetoric to describe this being developed in the 1960's that the Project resorted to recording actual classroom lessons on film and on audio-tape.

Probably no adequate rhetoric yet exists. There have developed a certain number of practitioner's maxims, generally quite similar to those of clinical psychology and analytically-oriented psychiatry.

It is hoped that the films will be used by scholars seeking to develop such a rhetoric, and at present the films are made available free of charge for this purpose.[71] How fruitful the approach via explicit abstract description will be during the second half of the twentieth century is precisely one of the main epistemological questions before us. It lies generally in the same area as other matters of *art* which may or may not be usefully discussed via explicit abstract symbolically-coded generalizations, and via media other than their usual media of presentation. The teacher in the classroom hears the overtones in children's voices, and sees their facial expressions. All good teachers believe they are guided by these cues, perhaps more than by any others, in making those decisions that a teacher makes in class.

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[71] Cf. especially the discussion in Shulman and Keislar, ed. (94).

One episode will be mentioned here. It occurs during the filmed lesson entitled Second Lesson, and may be viewed by any interested reader.

A third-grade girl named Ruth (the class itself being ungraded) describes a collection by saying "You can go on and on ... and ... and never stop!"

There is wonder in her voice, and a sense of excitement and comprehension.

A fourth-grade girl named Kate stands up and says: "The word for that is 'infinite'."

The teacher preferred Ruth's direct and seemingly honest language, and feared that Kate was leading the class toward the extreme peril of superficial verbal facility -- as the late Professor Raphael Salem said of M.I.T. freshmen: "Every freshman can tell you that the derivative of log x is one over x, but he doesn't know what a 'log' is, and he doesn't know what a 'derivative' is, and he doesn't know that he doesn't know."

Consequently the teacher indicated a preference for Ruth's way of saying it. At this moment Kate's face fell; her proud contribution had been ill-received (there may have been some status-incongruity involved also, since Ruth was younger). The teacher observed Kate's change of facial expression out of the corner of his eye, and -- in order to restore himself to Kate's favor -- he felt compelled to use Kate's word "infinite" rather prominently for the next few moments, although on cognitive and mathematical grounds he wished the word had not yet entered the discussion.

Madison Project teachers appear to be nearly unanimous in believing that the proper emotional "tone" to the classroom is essential if students are to turn in creative, superlatively original contributions. The desired tone might be described as one in which students feel considerable freedom, in which attention focuses on the task at hand, in which students allow themselves to become deeply involved, and in which students are not anxious.

The lengthiest verbal descriptions of various tape-recorded lessons were prepared by a panel who listened to various lessons by various teachers, and wrote descriptions of them.

The descriptions bear conspicuous resemblance to the language used by Eric Berne in Games People Play, although the lesson descriptions were written before Berne's book appeared in 1964. They also resemble Haim Ginott's descriptions of adult-child relations in his recent volume Between Parent and Child,[72] or any of a quite large school of modern thought concerning adult-child relations. It is interesting that few American classrooms would rate approval under the criteria most of these writers use. Cf., e.g., Brecher and Brecher (6), or Thomas (105),[72a] the latter of which contains such remarks as:

> A worthwhile form of discipline is permissive to the extent that it allows a child the freedom to explore, to discover, to learn through his own actions and mistakes.
>
> ... Love is one of the ingredients in this totality. Without love, none of the technique works.

This same tenor can be heard in the writings of Mearns (73), Boulle (4), deCharms and Carpenter (31), Ashton-Warner (2), Holt (50), Reik (79), Rogers (81), Kelley (59), and many others.[73] It sometimes seems implicit in Cremin (17). Yet at present no really explicit rhetoric exists to permit such discussions to free themselves from the level of "practitioner's maxims."

It is for this reason that the Madison Project has recorded actual classroom lessons on film, video-tape, and audio-tape.

The reader is asked whether he could use words to describe the difference between Haydn and Scarlatti so that a listener, unfamiliar with either, could now distinguish them. Clearly mathematical (information theoretic) methods for such a discrimination can probably be created. But would even these allow a person, having heard neither Haydn nor Scarlatti, to compose original music in the style of Haydn that would rise above the level of rather mediocre Haydn, "one of Haydn's lesser works?"

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[72] Ginott (42).

[72a] From: Rachelle Thomas, "A New Approach to Discipline," Parents' Magazine, Vol. XLII, No. 5 (May, 1967), pp. 59ff. Reprinted by permission of Parents' Magazine.

[73] Cf. especially Stern (99).

In the meantime, the use of actual musical instruments for the performance of Haydn, and the recording of such performances via audio technology, are clearly the method of choice for a basic presentation of the "results." The *discussion* of those results is usually in the rhetoric either of music critics, or of harmony, counterpoint, and orchestration.

To judge the classroom atmosphere of Project lessons, the reader is urged to view an appropriate selection of Project films.

D. Articulation of Practitioner's Maxims

The construction of appropriate "classroom happenings" is not entirely a matter of accident. Two guiding principles have been stated earlier, the "paradigm model of knowledge" (cf.p.80), and its corollary, the "do ... then discuss" strategy for teaching.

We give here two examples of how these principles have shaped two specific classroom experiences, and then proceed to one extrapolation.

1. *Inner products and matrix algebra:* "Candy-Store Arithmetic." (This is taken from "Curriculum ϵ," and is intended for able students.) There are many possible approaches to introducing matrix algebra,[74] among which the Madison Project has chosen to begin with a device suggested by Gerald Thompson, which illustrates excellently the "do ... then discuss" strategy of teaching.

The approach to multiplication of matrices is based (when one uses this line of attack) upon the concept of the inner product of a co-variant vector and a contra-variant vector, i.e.,

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[74] Cf., e.g., Cambridge Conference on School Mathematics (10), p. 53 and elsewhere.

$$(a \quad b \quad c)\begin{pmatrix} u \\ v \\ w \end{pmatrix} = (a \times u) + (b \times v) + (c \times w) \quad .$$

Clearly, as the above formula indicates, one could introduce "inner product" via the concept of variables, as indeed we have just done in the preceding formula. In the classroom work with fifth-grade children, we proceed quite differently, using Gerald Thompson's "Candy-Store Arithemtic."

The Thompson device has the virtue of causing the child, by himself, to compute correctly an inner product. The child has not analyzed it or thought of it in this way, but he has performed the actual act. The "thinking about it" then comes later.

The problem is presented with a story: you are in a candy shop that sells chocolate almond bars, peppermints, and chocolate-covered ants

$$\begin{array}{ccc} (& &) \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array} \begin{pmatrix} \\ \\ \end{pmatrix} \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} \quad .$$

The chocolate almond bars cost 10¢ each

$$\begin{array}{ccc} (& &) \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array} \begin{pmatrix} 10 \\ \\ \end{pmatrix} \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} \quad ,$$

the peppermints cost 2¢ each

$$\begin{array}{ccc} (& &) \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array} \begin{pmatrix} 10 \\ 2 \\ \end{pmatrix} \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} \quad ,$$

and the chocolate-covered ants cost 50¢ a box

$$\begin{array}{ccc} (\; & & \;) \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array} \begin{pmatrix} 10 \\ 2 \\ 50 \end{pmatrix} \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} .$$

Now, you buy three chocolate almond bars

$$\begin{array}{ccc} (\;3 & & \;) \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array} \begin{pmatrix} 10 \\ 2 \\ 50 \end{pmatrix} \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} ,$$

one peppermint

$$\begin{array}{ccc} (\;3 & 1 & \;) \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array} \begin{pmatrix} 10 \\ 2 \\ 50 \end{pmatrix} \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} ,$$

and zero boxes of chocolate-covered ants

$$\begin{array}{ccc} (\;3 & 1 & 0\;) \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array} \begin{pmatrix} 10 \\ 2 \\ 50 \end{pmatrix} \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} .$$

The teacher now says to the student: "How much money do you spend? Don't just tell me the answer, but show me how you work it out."

Recalling that this is part of "curriculum ϵ," intended mainly for able students, it is not surprising -- as has indeed been the case -- that students in nearly all cases will write:

$$\begin{aligned} \left(\begin{array}{ccc} 3 & 1 & 0 \\ \text{C.A.B.} & \text{P.} & \text{C.C.A.} \end{array}\right) \begin{array}{cl} \left(\begin{array}{c} 10 \\ 2 \\ 50 \end{array}\right) & \begin{array}{l} \text{C.A.B.} \\ \text{P.} \\ \text{C.C.A.} \end{array} \end{array} &= (3 \times 10) + (1 \times 2) + (0 \times 50) \\ &= 30 + 2 + 0 \\ &= 32 \ . \end{aligned}$$

At this point the teacher says (in effect): "Congratulations! You have just computed an inner product!"

This act having been performed, the teacher and students now discuss it in various ways. For example, they carry out similar calculations without benefit of any "story line" to guide them, etc., and this is ultimately expanded to matrix multiplication in general.

This example of a curriculum unit in the form of a very short "classroom happening" is intended to illustrate the "do ... then discuss" teaching strategy, and also hopefully to suggest the "paradigm model of knowledge" -- this experience of "Candy-Store Arithmetic" and the resulting visual blackboard display constitute the basic "paradigm" which the student is presumed to store in memory, with a capability of replay as if it had been recorded on film.

Marshall McLuhan's well-known remarks would seem relevant to this;[75] but earlier descriptions of the same sort were made by Professor Richard Alpert of Harvard University on the occasion of his viewing some Madison Project films showing actual classroom lessons.

2. "Guessing Functions" or "What's My Rule." Another lesson (or classroom happening) cast in nearly an identical mold was suggested by W. Warwick Sawyer. A group of three children work together, so as to be able to check one another's arithmetical errors. They "make up a rule" -- that is to say, something of the form "Whatever number you tell me, I'll double it and add seven, and tell you the answer."

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[75] McLuhan (68).

The class now tells the three children some number, the three "use their rule on this number and tell the answer," -- but the three do not tell what their rule is! By repetitions of the class telling a number, and the three telling an answer, the students generate part of the table for the function. Using "□" to represent "the number the class says" and "△" to represent "the answering number the three give," the students build up a table looking, say, like this:

□	△
0	6
1	9
2	12
3	15

Sooner or later the class are able to "guess the rule," to state it in words, and to write an algebraic formula to represent the function.

The preceding classroom activity provides the "paradigm" for the concept of function. It also represents the beginnings of a "do ... then discuss" teaching strategy. The situation is pregnant with possibilities for discussion.

The class may guess the rule as "whatever number we tell you, you multiply it by 3 and then add 6," and may write

$$(\square \times 3) + 6 = \triangle.$$

The three children who made up the rule may reject this, and argue that their rule was "whatever number you say, we add 2, then we multiply by 3," and write

$$(\square + 2) \times 3 = \triangle.$$

The ensuing discussion will then bring out the distinction between "formula" and "function," will lead to an agreement that guessing formulas is hopeless so that one must define the game in terms of guessing functions, and will possibly lead to the recognition of an *identity*:

$$\forall_{\square} \quad (\square \times 3) + 6 = (\square + 2) \times 3 \quad ,$$

for which, in "curriculum $\mathcal{E}$," the children might then write a correct derivation.

3. *Extrapolation: Can we design some classroom experiences to distinguish "additive" from "multiplicative" situations?* Polanyi's "practitioner's maxims" are unlike scientific generalizations in the respect that "practitioner's maxims" require an *art*, not fully understood in explicit terms, in order to be translated into effective use, whereas genuine scientific generalizations are sufficiently explicit that they could, in principle at least, be utilized by a properly programmed digital computer. Thus, F = ma and the *modus ponens* inference criterion applied to statements written in a specified form constitute scientific generalizations. The advice that a teacher should "not let himself come between the child and the mathematics" is a practitioner's maxim. Haim Ginott's advice that we draw the *meaning* of a child's statement from the over-all context, and not *merely* from the statement itself,[76] is presumably a practitioner's maxim.[77]

Whatever their differences may be, both practitioner's maxims and scientific generalizations have an extrapolation capability in design work: they enable us to *design* new artifacts, experiences, or whatever, that are not simply drawn directly from past experience.

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[76] Cf. Ginott (42), pp. 17 and 18.

[77] Probably the distinction drawn here between practitioner's maxims vs. scientific generalizations has a partially wrong focus, namely on the *explicitness* of the *instructions*. Perhaps an even larger difference lies in the degree of explicitness of the *anticipated outcomes*. Given sufficiently sophisticated programming, it is quite possible that a computer could in fact execute practitioner's maxims, but it might not know *precisely* what outcome variables to monitor.

To illustrate the two notions used in the preceding examples, we now use them to design a new set of classroom experiences that have not yet been used in trials with children.

Actual Project experience over the past eight years has indicated that children (for example, college-capable fifth and sixth graders) often confuse "additive" situations with "ratio" situations. The following is an actual example: Given the problem that a six-foot-tall man casts a four-foot shadow, and a certain flagpole casts a twenty-four-foot shadow, some bright fifth and sixth graders will argue that the flagpole must be twenty-six feet tall, on the grounds that the "thing" is evidently two feet longer than its shadow.

In working indoors at a chalk-board, the Project has never found any entirely satisfactory way to settle this question.

Lauren Woodby's elegant "outdoors mathematics" provides an exceedingly natural solution: children explore "how shadows really work" by driving stakes of varying lengths into the ground, and making a table relating their lengths to the lengths of their shadows, which might look like the following:

Length (or, if you prefer, "height") of stake above ground, in feet

Length of shadow, in feet

□	△
6	4
3	2
5	$3\frac{1}{3}$
$4\frac{1}{2}$	3
$1\frac{1}{2}$	1

One can now test the "additive" conjecture

$$\square - 2 = \triangle$$

against this table, and see at once that it is false. One can do more: since the children know the "Guessing Functions" or "What's My Rule?" game, they can use this method on the table, and obtain a formula to represent the function

$$\square \times \frac{2}{3} = \triangle$$

or, consequently,

$$\frac{\square}{\triangle} = \frac{3}{2} \quad ,$$

or even

$$\frac{\square}{6} = \frac{\triangle}{4} \quad , \text{etc.}$$

This provides an honest "process" answer for the specific question of shadows, which should presumably also be related to the similar triangles that this problem suggests.

But what is involved here seems in fact to be basically a problem in classifying functions, and hence the classroom activity that is desired is presumably one that involves the task of classifying various functions. These functions could come from a variety of sources, such as:

i) stretching of a spring vs. weights hanging on the spring;

ii) given a fixed loop of string, graph the area it encloses when it is shaped into a rectangle, vs. the height of the rectangle (used with children by Edith Biggs);

iii) given rectangles with a common base (not one unit in length), cf. area vs. height;

iv) height of child's head above the floor, as he stands on stools, chairs, tables, etc., of different heights;

v) area of triangles of fixed base as a function of their height (easily done on geoboards).

To serve as conceptually-useful "paradigms," one needs a few specific function situations that stand out clearly and that are emphasized dramatically in class. If we wish to minimize the number of function types, we could restrict attention to functions either of the type

$$\square + a = \triangle$$

or else of the type

$$\square \times a = \triangle.$$

To establish these two types as "paradigms" for subsequent reference, we need to emphasize:

i) the experience from which the function is obtained;

ii) the table of the function, including interpolation and extrapolation;

iii) the Cartesian graph of the function;

iv) an algebraic formula (in standardized form) to represent the function.

In subsequent lessons, as new instances of these functions arose, we would refer back to the original "prototypes" or "paradigms."

[Conceivably, if the idea of classification itself is uncomfortable for the students, one would use some *earlier* experiences in classifying physical objects -- *à la* Robert Karplus's SCIS Project -- and so on.]

Once these two standard types were firmly established in the child's mind, their role would probably be *strengthened*, rather than *weakened*, by subsequently moving on to establish many more categories of functions (quadratic, linear but not through origin, conic sections, higher degree curves, etc.), and children could make up their own methods for carrying out the classification (graphical, finite difference methods, etc.).

Thus, while our practitioner's maxims may help to shape some new curriculum units, there remains a need -- to which nearly all current curriculum projects can testify -- for extensive testing and modifying in actual classroom settings. Indeed, it is probably precisely this need for extensive testing of curriculum units that distinguishes the curriculum reform effort of the 1950's and 1960's from earlier "progressive education." The points of resemblance are exceedingly numerous, but whereas progressive education (perhaps unconsciously) assumed that a bright, creative, resourceful teacher, drawing on a course in freshman mathematics, a single course in chemistry, a single course in biology, etc., could work alongside her students and devise worthwhile classroom (or outside-of-school) experiences from which the children could learn about the contemporary world, today's curriculum movement makes almost an opposite assumption about means, while largely retaining the same goals. Today's assumption is that effective classroom experiences for young children should indeed by developed in the classroom, and with the children as junior colleagues rather than "students"--but the adults who shape the experiences must possess a very deep knowledge of the actual area of study.[78]

Summary: The "results" of this implementation project exist in actual classrooms in actual schools. What is contained here is merely a description of these results. A more complete record of the results exists in a more suitable medium: 16 mm. motion picture films showing actual classroom lessons. The seriously interested reader is urged to view these films.

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[78] Thus Jerrold Zacharias has pointed out that even physicists who have won Nobel prizes must sometimes sit around and think about the "real meaning" of some physics before they can devise effective learning experiences for children. This is quite a different assumption, indeed!

IV. *Discussion*

The project reported here was an implementation program in curriculum. This is a type of activity which was well-known in the late nineteenth century, and early twentieth century, in schools in the United States,[79] and has become prominent in England in recent years.[80] It has tended to disappear from the recent educational picture in the United States, or at least to diminish considerably in importance.

The purpose of this project was *not* to formulate, nor to gather evidence for, one or more symbolically-coded generalizations about classroom behavior. It was, instead, to *create* and to *exhibit* some *different* forms of classroom behavior, markedly distinct from those which are ordinarily observable in schools.

Thus its purpose relates to David Hawkins' remark that "To call something an independent variable is not to use a name but to claim an achievement."

It also relates to the well-known historical role of diversity. "Cultures" have blossomed especially when brought into contact with *different* cultures, through exploration, trade routes, or wars. Indeed, Erich Kahler (57) has argued that the word *culture,* deriving from *agricultura,* was essentially singular — meaning the only possible culture, that unique culture which *we* possess — until the researches and analyses of modern historians and anthropologists (as in the work of the nineteenth century Swiss historian, Jakob Burckhardt) gave it its present meaning of *a* culture, one of many possible cultures.

The reader of this report must, then, understand what he is reading: it is surely not "proof" of anything, and is only partially a description of something. Like psychotherapy or group therapy, the classroom lessons created by the Project exist as human experiences, and suffer drastically from translation into any other media. They were never destined for the printed page.

In today's world, however, effective dissemination does not depend upon the printed page. Many actual classroom lessons have been recorded on video-tape, on audio-tape, or on 16 mm. sound motion picture film. These films (which were not financed by the U.S.O.E., but rather by N.S.F. and other agencies) are the primary record of what has been done.

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[79] Cf. Cremin (17).

[80] Cf. Nuffield Project (55), and Featherstone (35) and (36).

A second form of dissemination has been used on a large scale, namely by having about forty teachers experienced in the teaching of the Project's various curricula fly into a major city (this program in fact began in Chicago under the leadership of Evelyn Carlson and Bernice Antoine), conduct an intensive two-week workshop for several hundred teachers in that city, and thereafter assist these teachers in conducting after-school and Saturday in-service courses. The films showing actual classroom lessons have played a central role in all of these workshops and in-service courses. This program in the Chicago Public Schools is now in its fifth year of operation, and similar programs operate in the New York City Public Schools, the Los Angeles City Schools, San Diego County Schools, and in the Philadelphia Public Schools.

Modified versions of such workshops have also operated in Darien, Connecticut, in Corpus Christi, Texas, in St. Louis, Missouri, in Washington, D.C., and in the area of Kansas City.

Thus the program has consisted of

i) creation of new classroom experiences with mathematics;

ii) "viability testing" of such lessons, by gross criteria of effectiveness;

iii) dissemination of some of those lessons which have survived the "viability testing";

iv) recording actual classroom lessons on film.

In all of this work, the Project has been, in effect, paying heed to Eric Hofer's remark that "history is made by example" -- indeed, various studies of the diffusion of innovations (even from ancient times) show that they are generally carried to new sites by the movements of actual human beings who have previously experienced them elsewhere.

The conventional wisdom of education has, in recent years, been based upon a number of assumptions quite different from those used in the present project. It is not our purpose here to dispute this conventional wisdom, but rather to call attention to the disparity, which may impede understanding.

i) The conventional wisdom has frequently conceptualized education as a "before" and "after" process that changes people, and has focussed attention on attempts to measure this change; the Project has focussed instead on the actual experience itself. In doing so it has accepted the conceptualization of David Hawkins (46), of Paul Goodman,[81] and of others, of "today's life for today's sake." Much can be said in defense of either conceptualization, but experiences outside of school are commonly viewed as life itself, to be valued for their own sake, and such a view may reasonably be taken toward experiences that are called "educational." Indeed, such a view becomes increasingly relevant as more and more people spend more and more time participating in "educational" activities. We cannot spend our lives waiting for tomorrow, or as Eric Berne has said, "for death or Santa Claus."

ii) Critics of modern curriculum innovations have argued particularly that the new programs do not include adequate "evaluation." In arguing this way, the critics appear to ignore the assumptions which underlie their own position. For one thing, they usually assume that the goals of education are known, that they can be stated explicitly, that they are matters of common agreement, and that measurements related to the achievement of these goals can be made more-or-less harmlessly. Yet for each of these assumptions an opposite assumption is no less tenable: probably no adequate list of goals exists in explicit form, with the consequence that relatively trivial goals which have been stated explicitly are greatly overemphasized, to the considerable detriment of all those other goals which are not explicitly stated; quite simple observation reveals that goals of education are far from being matters of common agreement; and there is no doubt that the act of making any measurements can be harmful: first, because teachers come to emphasize what they expect will be measured, and second, because that which is measured comes to be more highly regarded, and to contribute thereby greater stability to a part of our culture which is already far too rigidly inflexible.

This is not to say that measurements should never be made, any more than one would proscribe exploratory surgery, but in either case one should balance the estimated gain in information against the probable costs.

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[81] Cf. Schrag (88).

There are further reasons why curriculum projects have had little faith in "evaluation."

For one thing, no simple direct comparison with any other curriculum makes sense, because in nearly every instance the goals are different. A test aimed at one set of goals would ordinarily prove the superiority of that curriculum. (Indeed, the matter is far more subtle than even these words suggest, for matters of notation, definition, etc., vary from one curriculum to another. Hence questions that are not biased for or against some curricula are few and far between. For example, some authors consider that the symbol 4 denotes the same thing that the symbol $^{+}4$ denotes, while other authors consider that these symbols refer to quite distinct mathematical entities. This problem is far more severe than most observers seem to have realized. It is somewhat as if we taught spoken French in some elementary schools, spoken Spanish in others, spoken Russian in others, and spoken Chinese in still others. No simple direct comparison on a single test could prove the superiority of one of these programs. We could describe them -- e.g., can the child read an ordinary daily newspaper in the language he has studied; can he give instructions for finding one's way around town; and so on. But any hope that direct comparison could establish the superiority of one curriculum over another would seem to be based upon an inadequate conceptualization of the matter at hand.)

There are still many more aspects to the "evaluation" and "description" problems. For one thing, there is the question of whether "a curriculum" is properly thought of as something which is susceptible to "evaluation" in the sense of objective measurement, any more than the works of Beethoven are. What is more likely to be required are very much finer decision procedures, or measurement procedures, that can begin to determine which things, experienced in which ways, by which children, will produce which results. Thus, one may feel that Gieseking played Debussy with superb clarity, but somehow omitted the poetry and the philosophy from Beethoven, whereas Rubinstein achieves profundity in Beethoven but lacks clarity in Debussy. "A curriculum" will mean different things in the hands of different teachers, and will produce different results in different school environments.

As with so many comments in education, this is all quite obvious. Yet the theme persists -- it's not what you do, it's how you do it. Rosenthal has pointed out that global judgements such as "the subject responded with anger" seem often closer to the essential reality than more minutely specific discriminations, such as "the subject averted his eyes." If we measure what people do -- in the case either of teachers or of students -- and ignore how they do it, we run the risk of acquiring harmful misinformation.

It has recently been pointed out by one observer that sadistic or authoritarian teachers have in some instances taken the Project's "discovery" procedures and used them as weapons against children, by way of gratuitous denials of children's legitimate requests for help. This is no condemnation of "discovery" teaching; rather, it suggests that Beethoven emerges differently from the fingers of different pianists. So, also, with curriculum and pedagogy. It has long been a practitioner's maxim with clinical therapists that the therapist should not employ a procedure with which he feels uncomfortable. Some similar remark undoubtedly applies to teachers. All of this tends to destroy any concept of "the curriculum" as viably invariant.

It would not be true, however, that the Project "uses no evaluation." Quite the contrary; it makes a great deal of use of feedback data, often of quite subtle sorts.

In the first place, a "process" approach means that the teacher does not give a lecture, thereafter being left to wonder about what the students heard. The situation is entirely different; the child has actively done something -- performed a measurement, stated a generalization, constructed the proof of a theorem, etc. -- while the teacher watched. Thus the teacher is not left in doubt as to what the child can do.

The Project's recent emphasis on small-group and individualized instruction carries this even further: the teacher has more opportunities to observe student behavior, and to observe every student.

Thus the teacher is getting a generous helping of feedback information on student performance.

In the second place, the classroom lessons used are exceptionally fully-worked-out. Thus curriculum and pedagogy are combined, just as plot, dialogue, and stage directions are in a play, and the teacher is given enough information about this combination -- for example, via film -- to be able to teach the lesson in fairly "authentic" form. The what is then taken care of; future attention could quite properly focus on the how -- the nuances of tone of voice, the gestures, the movements about the room, by which the teacher supports, threatens, cajoles, or gives cues to correct answers.

In the third place, there are the films themselves. Many actual lessons can be viewed just as they occurred, with all movements, nuances, etc., intact.

In the fourth place, the Project operates in a goldfish bowl; literally thousands of teachers have observed classroom lessons and subsequently discussed them; hundreds of professionals of various sorts have observed lessons and commented upon them; many dozens, and perhaps by now a few hundred teachers have themselves taught lessons which were observed and discussed by others.

Finally, in the fifth place, the Project has reported as fully as possible to some of the most competent professional groups in the nation (and, in fact, also in England, Canada, Mexico, Australia, Japan, Sweden, Hungary, Ghana, Uganda, and the Soviet Union), often again using films to show exactly what the children are doing and how they are doing it.

During the past seven years the Project has made such reports to: various meetings of Educational Services, Inc.; the "Learning About Learning" conference at Harvard;[82] the African Mathematics Program of E.D.C.; a meeting of the American Mathematical Society at Miami, Florida; a presentation at the United States Office of Education Demonstration Center in Washington, D.C.; meetings of the Cambridge Conference on School Mathematics; the Ditchley Conference in England; the conference on "Discovery Learning" held by the Social Science Research Council;[83] a meeting of the A.E.R.A.; the Piaget meetings at Cornell University and at Berkeley;[84] and various meetings of the National Council of Teachers of Mathematics in the United States, and of the Association of Teachers of Mathematics in England. In all of these meetings, the emphasis has been on a dialogue about the content or methodology of the Project's work. As a result the Project has enjoyed a large amount of guidance, in addition to that provided by the staff or consultants employed by the Project.

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[82] Cf. Bruner (7).

[83] Cf. Shulman and Keislar (94).

[84] Cf. Rockcastle and Ripple (80).

Perhaps the main theme of the present discussion is that the armamentarium of methodology for the advancement of education must be enlarged. Elaborate empirical comparisons of alternative curricula have at best a small role to play -- perhaps a negligible one. There is urgent need to create a far wider diversity of curriculum-cum-pedagogy "experiences," and an equally great need to develop more ways of conceptualizing the entire undertaking.

The classroom lessons developed by the Madison Project, and recorded on film, will be made as available as possible to any who are interested in pursuing this matter. These lessons have settled virtually nothing, but they may help to start something worthwhile.

V. Conclusions, Implications, and Recommendations

Viewing "curriculum" and "pedagogy" as two aspects of an indivisible unity, the Project has been concerned with developing sequences of classroom lessons combining mathematics and science at the level of grades K-9, with the major emphasis being on mathematics. The Project has further sought to work with various teachers and school systems in implementing this program in actual classrooms.

In such a relationship, the Project maintains its separate identity (and its independent goals and opinions), but the curriculum which emerges will be in each case a compromise between the Project and the school.

The broad Project goal would be an elementary school organized after the fashion of the British "integrated day," in which no sharp lines of demarcation separate the various subjects; there would be great diversity of multi-sensory experiences for the children, arranged flexibly, but in the service of a carefully planned cumulative "curriculum."[85]

The schools with which the Project has cooperated have necessarily always shared some goals with the Project, but have probably never had identical goals. Cooperating schools have included the public schools of Weston, Connecticut; Scarsdale, New York; Lexington, Massachusetts; Greece, New York; Ladue, Missouri; University City, Missouri; Elk Grove, Illinois; St. Louis, Missouri; Washington, D.C.; Chicago, Illinois; San Diego County, California; Los Angeles, California; New York City; and Philadelphia; plus some other public schools, and such private schools as Nerinx High School in Webster Groves, Missouri, and St. Thomas Choir School in New York City.

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[85] Cf. Featherstone (35), and (36); one of the best descriptions of the Project's general vision of an "ideal" school has been written by Peter Shoresman of the University of Illinois, as a "preliminary working paper" of the E.D.C.-sponsored Cambridge Conference meeting at Pine Manor Junior College in August, 1967; cf. also the Nuffield Mathematics Project booklet I Do ... And I Understand, and the film of the same name. Part of the Shoresman essay is reproduced, with permission, as Appendix H of this report.

From this collaboration there have emerged a variety of different curricula, five of which are discussed at some length in the body of this report, namely:

"Curriculum α," intended as a supplementary program for grades 2 through 8, with the purpose of providing a foundation for a program combining arithmetic, algebra, geometry, and science. This curriculum is basic, and can be used in severely stressed urban settings.

"Curriculum β" is an extension of "α," with more arithmetic, more science, a greater diversity of multi-sensory experiences, and with far greater potential for matching the experiences to the needs of the individual child.

"Curriculum γ" is a simplified program for children in nursery school, kindergarten, and grades 1 and 2. (Cf., e.g., Appendix E for a sample.)

"Curriculum δ" is a ninth-grade course for college preparatory students. It provides a suitable continuation for students who have studied "modern mathematics" programs in earlier years, but it does not necessarily presuppose any prior background in "modern mathematics."

"Curriculum ϵ" is a sophisticated program for gifted students, in grades 3 through 8. It can also be adapted for use with other children. It is distinct from programs α, β, and γ in that these first three programs work realistically in ordinary classroom situations, including urban slums. By contrast, "curriculum ϵ" appears to work successfully only when conditions are "just right." The Project has been unable to determine what "just right" really means. This curriculum is highly suggestive, but would not be a practical answer in most school settings. (Incidentally, "cultural deprivation" as usually defined at present does not seem relevant; "ϵ" has worked satisfactorily with privileged children, and also with "culturally-deprived" children; but it does not work reliably in random situations.)

For most of these programs, actual classroom lessons have been worked out in relatively precise detail, and (once the lesson is perfected) it is taught to a new class of children, for whom it is a new lesson, and this lesson is recorded on videotape with subsequent transfer to 16 mm. film, or else it is recorded on audiotape, usually with a two-channel recorder (in an approximately binaural fashion).

These films and tapes were financed by agencies other than U.S.O.E.; they will be made available to the academic community as freely as possible.

The various curricula have not depended upon textbooks -- indeed, the Project's view of the ideal elementary school would not have the curriculum structured by a textbook series at all, but by activities selected carefully by the teacher. Such schools presently exist, especially in England.[86]

Although these curricula and these various lessons do not settle very much, they can be quite provocative. This raises some very serious considerations about appropriate means of dissemination, which will for the most part be considered in the "Recommendations" section of this report.

One means of dissemination needs to be mentioned here: these are the "big city" workshops, operated by the Madison Project, but in fact designed by Mrs. Evelyn Carlson and Mrs. Bernice Antoine of the Chicago Public Schools, and subsequently modified and utilized by Dr. John Huffman, Dr. Jack Price, and John Gessel of San Diego County, by George Arbogast of the Los Angeles City Schools, by George Grossman of the New York City Schools, and by Karl Kalman of the Philadelphia Public Schools. A more detailed description of these workshops can be obtained by writing to the Madison Project, 8356 Big Bend Blvd., Webster Groves, Missouri 63119.

The best way for the serious reader to comprehend these curricula and lessons is either to view a number of the films, or else to attend one of the Project's workshops.

The "flavor" that is intended throughout is one which minimizes (in Schwab's words) "a rhetoric of conclusions," and which maximizes "a rhetoric of inquiry." As discussed earlier in this proposal, the curricula and lessons seek to shift away from over-emphasis on "facts," and to put far greater emphasis on such processes as measurement, making simple abstract models for complex real situations, conjecturing, proving, extending, explicating, refuting, discovering patterns, "exploring," developing problem-solving strategies, and so on.

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[86] Cf. Davis (28).

This flavor is presently absent from most mathematics instruction at the pre-college level, even in 1967. Yet these are natural activities of children, and are essential for resourceful, flexible, wise adults. We must see that knowledge is not allowed to crowd out wisdom.

Recommendations.

A look at the successes and failures of a program as old and as large as the present one is ipso facto a look at a sizeable portion of contemporary education in the United States. Thus, the successes and failures of the Project, while not extremely important in themselves, acquire great importance as general problems of our educational system. The reader is asked to forget the specific instance of the Madison Project, and to think of nearly any educational problem which presently needs solution -- conceivably, for example, teaching Americans facility in a second language, or teaching a deeper understanding of our society to a larger proportion of our citizens.[87]

Creating the Innovation. The "ideal" school of the "Progressive educators" looked amazingly similar to the "ideal" school of many of today's curriculum innovators. Both imagined children working freely, in groups, at self-selected tasks from which they were learning about themselves, their physical environment, their social environment, and their cultural heritage. One striking difference appears when we look at how these innovative programs were to be created. Progressive education usually implicitly assumed that a bright, creative teacher with a general education -- six credits of college mathematics, three credits of chemistry, three credits of economics, no industrial design, and no pediatrics nursing could work together with a class of ten-year-olds (who would be junior colleagues rather than students) to build (say) a miniature sulphuric acid factory, and thereby learn about themselves, their culture, its technology, the nature of cooperative human ventures, etc.

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[87] Incidentally, it should become clear that "Progressive education" and recent "curriculum reform" have a great deal in common, and are becoming more alike with each passing year. Consequently, in what follows, it is useful to ask: how is this different from "Progressive education"?

By contrast, the curriculum evolution movement of the 1950's and 1960's made entirely different assumptions about how an innovative curriculum unit can be created. A specialist with doctoral-level competence in the relevant subject area might work, more-or-less full time, for as long as two or three years designing an appropriate classroom unit.

This is the difference between a DC-3 and a 727.

Many of the units developed by today's curriculum innovators would, one hopes, have been acclaimed as triumphs by Colonel Parker or by John and Evelyn Dewey. But the effort that goes into designing them is orders-of-magnitude greater.[88]

While the creation of further curriculum-lesson experiences is still necessary -- indeed, it is a *sine qua non* of further real progress -- it is reasonable to look at the results of various projects and individuals (but, interestingly enough, not of commercial firms) and to conclude that progress probably will be made. In somewhat oversimplified terms, if the "creation of innovations" problem is not solved, it is on the way to being solved.[89]

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[88] As one example, in a unit on gases that was in the preliminary stages of development at the Cambridge Conference meeting in the summer of 1967, questions arose about comparing oxygen, helium, air, and carbon dioxide. One question dealt with the ability of oxygen to make a candle burn more brightly, but it was not an easy task to be sure that the candle *was* burning more brightly. How, then, to measure the brightness of a candle flame? Another task involved filling balloons with Co_2 vs. filling them with air, dropping them simultaneously, and seeing if one fell faster than the other. This involved some rather complex questions of aerodynamics, plus the possible use of statistical methods to confirm the reliability of this test. It also turned out -- unexpectedly -- that Co_2 -filled balloons deflated faster than air filled balloons. Any reasonable child is going to ask "why?" All right -- why?

Dissemination. Once an innovative curriculum unit or lesson has been created, it requires dissemination. An earlier age would have utilized textbooks for this purpose, but the spirit of many contemporary programs appears to be irreconcilable with standard textbook series. The "ideal" school would contain many pamphlets, books, workbooks, problem sheets, assignment cards, etc., but would de-emphasize (or completely abandon) any basic textbook, at least for grades K-6.

Four alternative dissemination schemes deserve consideration:

i) The "each-one-teach-one" approach. This is used, for example, in the Madison Project's intensive two-week workshops, after-school in-service courses, and Saturday in-service courses, although the actual ratio is more likely a two-man team teaching a class of thirty participants, and not "one" to "one."

ii) Recording actual classroom lessons on film. The purpose of these films is not to teach children, but rather to show a teacher how to conduct the lesson in question.

One can imagine a teachers' preparation room with a cartridge projector (such as the Fairchild Mark IV) and a stack of cartridges, where teachers could study a lesson before attempting to teach it -- but we have yet to see such a facility in actual operation.

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[89] Here it is important to note that Harvard University recently awarded a doctorate (to Marion Walter) for the creation of a significant curriculum unit -- on which she worked for three years. This form of doctoral thesis needs to be considered by other universities, as well.

iii) The English Nuffield Mathematics Project, led by Geoffrey Matthews, is in fact a non-textbook program of the type which the Madison Project favors. The Nuffield Project *does* prepare books *for teachers,* but not for students (again, we are speaking of grades K-6 or so). Instead, the Nuffield Project approaches the dissemination problem by creating "teacher study and preparation centers," located in school buildings dispersed geographically over England. The typical "center" contains a seminar room, comfortable chairs, facilities for making tea, and a room filled with appropriate sample curriculum materials, teacher reference books, etc.

In our own case, were the Madison Project able to design the "ideal" center of this type, we would take pains to make the process of relating mathematics to science, carpentry, etc., as simple as possible, and our ideal center would probably contain machine tools (such as drill presses, power saws, etc.), laboratory equipment (Bunsen burners, glassware, sinks and water taps, some chemicals, etc.), along with such obvious things as books and reports. It would (speaking still of the "ideal" Madison Project version) contain cartridge projectors and film cartridges showing actual lessons, perhaps a computer terminal, possibly a teletype machine connected to a central computer as in Patrick Suppes' program, *and it would also include a professional staff,* consisting perhaps of a machinist-glass blower type of person (of the kind the Elementary Science Study of E.D.C. puts to conspicuously good use), a professional in the area of science, mathematics, and teaching, and possibly such other people as: interns, graduate students, student teachers, paraprofessionals, classroom teachers, parents, and clerical assistants.

Teachers might visit centers individually or in groups, they might reserve a regular time each evening (as is done in allocating facilities at bowling alleys, public picnic grounds, etc.), they might arrange for specific lectures by designated experts, they might merely browse, they might build a specific piece of apparatus to take back to their own schools, they might check out equipment as one checks books out of a library, etc.

It may be argued that such institutions already exist in America. We have yet to see one. The center described here would be informal, friendly, creative, flexible, responsive to teacher-initiated suggestions, adequately equipped (and expecting its equipment to be *used,* sometimes lost, sometimes broken -- contingencies it would be able to take in stride, without trauma), in close contact with curriculum innovation projects, and so on. Perhaps most conspicuously, it would combine some profundity of knowledge with creativity and a determined sense of mission.

iv) The Madison Project's "ideal" elementary school, while by no means a well-formed design at present, is probably most adequately described by Peter Shoresman, in a working paper of the 1967 Cambridge Conference, part of which is reproduced here in Appendix H. One key feature would be that the sequence of activities would not be structured by a textbook series. It seems to the Project that contemporary use of textbook series puts rigidity in exactly the wrong place, and makes it virtually impossible for the teacher to deal with individual children as individuals, or to make the best of instructional opportunities.

Where would the structure of the program come from? A combination of sources: actual teacher expertise, from handbooks which would serve the teacher in much the same way that books on "Current Perspectives in Gastroenterology," "Research and Clinical Studies in Headache," or "Calcium Metabolism and Bone Disease" serve physicians, from curriculum units, films of sample lessons, and other devices suggested earlier -- and probably also from some specific institutions designed to provide assistance to teachers and to schools, in somewhat the way that the Madison Project has attempted to do for the past seven years or so.

The Madison Project as a Prototype Institution; the Problem of the "Old" Innovation. The Madison Project began essentially as a curriculum innovation project, and also to provide certain consulting services to schools. Over approximately the past decade its role has shifted; it has come to do less original creation of curriculum innovations, and to involve itself more heavily in providing services to schools. That it could undertake to operate workshops for literally thousands of teachers in school systems the size of New York City, Chicago, and Los Angeles is an indication of how heavily it has become involved in providing services to schools.

Conducting workshops is not the only such service; the Project also assists in planning curriculum and instruction, and has helped to create a college undergraduate program for the education of specialist teachers in elementary school mathematics (at Webster College), and a corresponding Master's Degree program (also at Webster College) whereby practicing elementary school teachers may become mathematics specialists.

In order to provide these services, it has been necessary to go beyond curriculum innovations which the Madison Project could create, and to "assemble" curricula from a variety of sources.

In doing this the Project has come to play a variety of roles which similar projects also play to varying degrees, but which are not provided for in the usual roster of educational institutions and facilities. This is, for the next few decades, probably the most significant fact of the entire "curriculum evolution" movement.

From among the entire array of curriculum innovation projects that have overlapped the present area of concern, at this point of time it appears that only three have achieved the stability of viable continuing organizations: the MINNEMAST Project under James Werntz, E.D.C. (formerly E.S.I.) in Newton (and Watertown), Massachusetts; and the U.I.C.S.M.-PLATO-CIRCE group at Urbana, Illinois. These last two have surely been the two most important organizations in the field, and are about the two oldest.[90]

What should the attributes and activities of a curriculum innovation organization be?[91]

i) Presumably it would develop new curriculum units (probably in the sense of curriculum-combined-with-pedagogy, i.e., in the sense of developing actual classroom lessons by extended preliminary trials in actual classrooms);[92]

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[90] Cf. Koerner (63).

[91] Some of these attributes are taken from an impressive list made by James Werntz for the 1967 Cambridge Conference meeting at Pine Manor; others have been suggested by Jack Easley of UICSM.

[92] This is quite different from the "writing group" approach in which the point of intervention is the textbook rather than the actual classroom lesson; also, in the "classroom" approach actual children are present and help to form the curriculum units, whereas in the "writing group" approach materials are prepared in advance, and possibly submitted for classroom trials after they have been written.

ii) It would operate (as the Madison Project has done with its "big cities" program) a kind of small-scale specialized "national teacher corps" -- for the purposes of conducting workshops, etc., it would be able to mobilize dozens of genuinely superior teachers who would possess specialized experience in the relevant areas of innovation, and would be experienced also in conducting precisely this sort of workshop (which may involve uses of closed-circuit TV, etc.);

iii) It would go beyond the boundaries of its own creations, and assemble a larger variety of relevant lessons than any single project can create, by making arrangements with other projects, etc. In assembling a "composite" program of this sort, the "neutral" organization (i.e., one which does not create any of its own original materials) is, for some reason, at a marked disadvantage vis-a-vis the "committed" organization which actively develops its own curriculum materials. The assumption that "neutral" organizations can "combine the best" from the several innovative organizations is contradicted by actual experience. One of the "innovative" groups is better able to engineer a synthesis of several programs.

iv) It would operate workshops for schools, and some, at least, of these "curriculum innovation organizations" would be prepared to do this on a very large scale; one is not talking here about a "workshop" to be attended by thirty-five participants from a school system (although these are valuable in the case of small school systems) -- for larger cities, one is talking about a workshop for six hundred participants, and even larger ones may exist in the future. At present no major university operates such a program (although several of them are possibly preparing to do so).

v) It would provide opportunities for significant contributions -- in the form of doctoral theses or otherwise -- by university graduate students, and would thereby go far toward reinstating curriculum and instruction as two of the foundations of education (a position from which they had earlier been unjustly -- or, at least, prematurely -- deposed by history, sociology, and psychology).

vi) In order to have any real effect, it would involve itself deeply in the undergraduate education of prospective elementary teachers or elementary school specialists, perhaps involving a unified offering in the area of mathematical content, "methods," and supervised teaching experiences.

vii) It is already clear that idealistic curriculum innovation projects can pursue elegance and ingenuity in such a way as to allow themselves to lose virtually all contact with the quite realistically grim and sordid situations of many of our schools.[93]

The schools, on their side, are often willing to live for far too long with situations which they should, indeed, reject as intolerable.

One solution to this problem, which E.D.C. is already beginning to utilize, *is for the innovative organization to assume the actual responsibility for designing and operating one or more schools.* This pattern could be mutually advantageous to everyone involved.

The curriculum evolution efforts of the 1950's and 1960's consequently seem to show two outstanding facts:

i) The effort has not quite been great enough, nor sustained long enough; just when schools and colleges are becoming fully ready to reach out and take advantage of innovations, the innovative organizations themselves are beginning to disappear. Effective innovation is not a "package" which is "produced" and "consumed"; it is more like dentistry, depending upon someone who desires a service matching up with someone who can provide it; specifically, a school or college seeking help in effecting an innovation must come into contact with an innovative organization that is able to provide such help. Furthermore, both organizations must command the necessary resources in materials, space, personnel, and so forth, to get the job done. *Probably this is a continuing need that will not disappear in the foreseeable future.*

ii) One is confronted, in effect, by a jigsaw-puzzle type of situation, in which the road to progress almost certainly depends upon fitting together a number of separate pieces. For example, if "curriculum creation centers" on the Nuffield plan are to be developed (cf. Featherstone (34), p. 15), where teachers could *themselves* participate in the creation of new teaching materials, these centers could be related to a *college that educates* teachers (and would relate both to the undergraduate program of the college, and also to "research" interests of college faculty), they should probably also be related to an innovative project (such as U.I.C.S.M., the Madison Project, E.D.C., etc., for a source of further ideas, personnel, and broader contacts), and they would be specifically related to one or more school systems (where they would serve most directly to introduce new curriculum-and-pedagogy ideas to teachers and curriculum planners).

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[93] Cf., for example, Smith and Geoffrey (97), and also *The National Observer* (104), p. 1.

Various universities already have, or are now creating, "Ph.D. in College Teaching" programs, or other similar programs. Theses in such programs often can include actual "curriculum innovation contributions," and are not restricted to "behavioural science" theses. (The obvious analogy is to the acceptance of either a creation -- a play, novel, etc. -- or else a "scholarly study" -- say, of the quarto editions of Shakespeare -- as suitable theses.) Theses of both types might be done in relation to the Nuffield-type "centers," thereby helping to meet the needs of the centers, and at the same time meeting needs of the doctoral programs. Such pieces have not fitted together well in the recent past, but there is every reason to believe that they could be arranged so that they would fit together in a mutually helpful way. Accomplishing this would provide a major step forward.

VI. Summary

Over the past decade the United States has seen several approaches to the improvement of the school program in science and in mathematics. The present effort has dealt with a small amount of science, but mainly with mathematics; the actual grade levels of the students have ranged from nursery school through grade 10, with the major focus on grades 2-7 or thereabouts. The material content -- as distinct from the children -- has included matrix algebra, statistics, geometry, logic, mechanics, and other topics, and therefore the content may be regarded as relevant also to older students (indeed, portions of this same program have been used at the college undergraduate level and in a Master of Arts in Teaching degree program, as well as in the in-service education of teachers).

The program developed has two important characteristics:

First, it operates at the level of actual classroom experiences of children, as distinct from, say, the level of preparing textbooks. Indeed, the program is not based upon textbooks: it is a "non-text program" of the type often seen in England (cf. Featherstone (34)).

From this point of view, the task of the Project has been to work with teachers in developing suitable classroom experiences for children, to train additional teachers in the use of these lessons, to test the appropriateness of the lessons, and to propagate the program more widely by operating workshops for other teachers, and also by recording typical lessons on film. This film program was supported by the Course Content Improvement section of the National Science Foundation, and the major workshops have been supported either by school systems or else by the Cooperative College-School Science Program of N.S.F.

The verification of the suitability of these lessons has been carried out through careful observation by mathematicians, teachers, administrators, and clinical psychologists; by "viability" testing in the hands of a variety of teachers of varying qualifications, and with a wide variety of students; by confirmation of appropriateness through viewing of films by relevant panels of professionals; by tape-recording lessons by a variety of teachers and allowing a panel to study these recordings; by following the same students for up to five years in the program in order to observe cumulative effects; and by tape-recorded interviews of students conducted by a clinical psychologist.

Second, the program is a supplementary program designed to allow schools to make certain specific modifications in their school mathematics programs, namely:

i) to provide a foundation for developing a K-8 program that unifies arithmetic, algebra, geometry, and some science;

ii) to shift the "tone" or "emphasis" of a school program away from a "rhetoric of conclusions" and toward a "process" approach;

iii) to move toward a greater use of physical materials and multi-sensory experiences in mathematics classes;

iv) to create greater opportunities for small-group work and individualized instruction;

v) to utilize a specific teaching strategy that is based upon a "do something...then discuss it" approach, sometimes also preceded by a period of exploration or "free play"; this is in contrast to the more usual verbal approaches that are based upon using the English language to tell students what to do, and to tell them how to do it;

vi) to create a more receptive environment for student initiative, especially where unexpected (but correct) responses are made by students;

vii) to open the door to a re-consideration of the grade-level placement of many topics;

viii) to open the door to a non-graded program;

ix) to make available the simplest possible program for students who are not experiencing success with mathematics;

x) to make available a more sophisticated and more advanced mathematics program for those students who can benefit from it.

These may, or may not, be objectives of a given school system; but if they are, the Project program represents a modest step in these various directions; the films exist and are available;[94] the Project has a group of experienced teachers who can conduct workshops for teachers; and -- thanks to the films showing actual classroom lessons -- teachers, supervisors, curriculum planners, etc., can view these films and decide upon the relevance of the program to their needs.

Looking more closely at the Project's work over the past decade, there are in fact a number of distinguishable "programs" that can be identified. Five are discussed in the main body of the present report. They might be described very briefly as:

"Curriculum α": A basic curriculum, primarily for grades 2-8 (although use with older students is feasible), intended to provide the basic lessons necessary to begin to unify arithmetic with algebra, geometry, and physical science;

"Curriculum β": This is an "assembled" curriculum, consisting of original Madison Project lessons (actually, "Curriculum α") combined with lessons developed by other Projects and individuals, especially by Marion Walter, by Lauren Woodby, by Leonard Sealey, by Z. P. Dienes, by Edith Biggs, by the Elementary Science Study of E.D.C., and by the English Nuffield Mathematics Project directed by Geoffrey Matthews. It differs from "Curriculum α" in that "Curriculum β" places greater emphasis on arithmetic, greater emphasis on science, involves a variety of approaches to geometry, makes more use of physical materials, and is designed for more emphasis in small-group work and individualized instruction.

"Curriculum γ": A simplified curriculum for nursery school, kindergarten, and grades one and two. (For some idea of this curriculum, cf. Appendix E to this report.)

"Curriculum δ": A ninth-grade course for college-capable students. (For a complete description of this course, see Davis (22).)

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[94] For information, contact The Madison Project, 8356 Big Bend Blvd., Webster Groves, Missouri 63119. (Telephone: Area Code 314, WO 2-0440.)

"Curriculum ϵ ": A "sophisticated" program for grades 3-8, that has been used successfully with "culturally privileged" children and with "culturally deprived" children -- hence "cultural deprivation" as usually estimated does not seem crucial. However, this program lacks "stability": it works successfully for some teachers, for some classes, and in some schools. More frequently it does not work successfully. The Project has been unable to determine the crucial differences between situations where this program works well, as against those where it does not.

It should be emphasized that Curricula α, β, and γ are intended for all children, and have been tested with a wide variety of children. Curriculum δ (the ninth-grade course) is for college-capable children, whether or not they have had "modern mathematics" in grades K-8 (and it has been tested with both groups). Curriculum ϵ is for children who do well in "discovering mathematics" -- these are not always children who were previously doing well in school, and there is no decisive relation to urban vs. suburban children, or "deprived" vs. "privileged," as usually determined.

Besides developing and propagating actual classroom experiences, as discussed above, the Project has also attempted to articulate a (rather primitive) "theory of instruction," or a set of "practitioner's maxims," of which possibly the most important is the teaching strategy of "doing something, and then discussing the experience afterwards." Examples are given in the present report, and can be viewed on the films. This teaching strategy is not commonly used in traditional mathematics teaching.

Where necessary, this "do ... then discuss" strategy is preceded by a period of exploration or "free play." Games, and various other tasks, may also be used to provide subsequent practice.

This strategy for teaching, possibly alongside the Project's decision procedures for selecting and shaping curriculum experiences, may represent the Project's most important contribution. These are, however, genuinely "practitioner's maxims" (in the sense of Polanyi (77)); they are not scientific generalizations, and would be largely vacuous and inane were it not for the large body of instances that have been created which serve to illustrate their meaning in terms of actual use. (Cf. Davis (21), (24), and (29).)

In addition to its work in schools, the Project has assisted in developing an undergraduate college program, at Webster College, that is intended to educate elementary school mathematics specialists, and has assisted in developing a Master of Arts in Teaching program (also at Webster College) that serves the same purpose for those who are presently elementary school teachers or administrators.[95]

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[95] Cf. Davis (28).

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Appendix A

Some Studies of the Effectiveness of Madison Project Materials.

A large variety of different methods have been used in studying diverse aspects of the effectiveness of the Project's various prototype lessons and lesson sequences. Some of these have been discussed within the body of this report. The present Appendix reviews some of these and introduces others.

The conventional wisdom of education often seems to repeat "Evaluate, evaluate" until the discussion begins to sound like a chorus by Tom Lehrer. The very act of repetition of a slogan with seemingly little question must itself raise a question. Indeed, there is enough unquestioning faith in the conceptualization of education as a "before" and "after" subject which somehow changes the student, in the possibility of an explicit statement of something called "goals" or "objectives," in the possibility and desirability of measurement, and in the value of something believed to resemble the "generalizations" of mathematics (or the quite different "generalizations" of physics) that one is compelled to ask why these notions are, indeed, accepted with so little scrutiny. Writing on a different but related subject, the eminent French anthropologist Claude Lévi-Strauss has contrasted "magical thought" with "scientific thinking":

> This preoccupation with exhaustive observation [of plants and animals, as undertaken by allegedly "primitive" peoples] and the systematic cataloguing of relations and connections can sometimes lead to scientifically valid results. The Blackfoot Indians for instance were able to prognosticate the approach of spring by the state of development of the foetus of bison which they took from the uterus of females killed in hunting. These successes cannot of course be isolated from the numerous other associations of the same kind which science condemns as illusory. It may however be the case that magical thought, that 'gigantic variation on the theme of the principle of Causality' as Hubert and Mauss called it (R. Hubert and M. Mauss, "Esquisse d'une théorie générale de la magie," L' Année Sociologique, Vol. VII, 1902-3; reprinted in M. Mauss, Sociologie et Anthropologie, Paris,

> 1950, p. 61), can be distinguished from science not so much by any ignorance or contempt of determinism but by a more imperious and uncompromising demand for it which can at the most be regarded as unreasonable and precipitate from the scientific point of view.
>
> . . .
>
> Seen in this way, the first difference between magic and science is therefore that magic postulates a complete and all-embracing determinism. Science, on the other hand, is based on a distinction between levels: only some of these admit forms of determinism; on others the same forms of determinism are held not to apply. One can go further and think of the rigorous precision of magical thought and ritual practices as an expression of the unconscious apprehension of the truth of determinism, the mode in which scientific phenomena exist. In this view, the operations of determinism are divined and made use of in an all-embracing fashion before being known and properly applied, and magical rites and beliefs appear as so many expressions of an act of faith in a science yet to be born.[96]

This passage is extremely suggestive for the study of education, and might reasonably serve -- alongside many parallel remarks of other authors -- to raise questions concerning the best way to understand education and to improve it. Who is being "scientific," and who is being "magical"? Is the wisdom of the artist-practitioner to be dismissed as "mere folk-lore," when so large a part of human society is so clearly based upon the valid portions of folk-lore, and still bedeviled by contamination from the non-valid portions?

In any event, the conventional wisdom of educational research has not proved especially advantageous to most curriculum projects of the past decade, and the Madison Project is no exception. The Project's own view is that, much as winter follows summer and is in turn followed by another summer, there are times when an approximate statement of goals will serve to clarify purposes and improve communications. Such times are followed by an era of activity and changes in which the actual goals are not fully articulated and only dimly perceived. These times give way to yet other times, when a new approximation to an explicit statement of goals becomes desirable. This is not the view of the conventional wisdom.

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[96] From: The Savage Mind by Claude Lévi-Strauss (66), pp. 10-11. Reprinted by permission of the publisher, University of Chicago Press.

At the present time (October, 1967), the Madison Project materials can be used to serve at least these goals -- although not all of them at the same time, and only by selecting properly from the Project's armamentarium:

i) building an improved understanding of certain commonplace topics in arithmetic, such as place-value numerals, algorithms, fractions, etc.;

ii) arousing an interest in school (or in mathematics) among children (and, for that matter, teachers) who have not lately exhibited a very lively involvement or an eager enthusiasm (this includes elementary education majors who believe they hate mathematics, etc.);

iii) providing a basic foundation for unifying arithmetic, algebra, and geometry in grades 2 through 9, or so;

iv) providing a basic foundation for relating mathematics to science (and even to such subjects as history and music, or so);

v) providing a program to allow more talented children to move ahead more quickly in mathematics;

vi) providing materials and ideas which enable teachers to change their mathematics classes from a textbook-dominated approach to a more flexible "mathematics laboratory" approach, including small-group work and individualization of instruction.

In point of fact considerable data has been gathered testifying to the proposition that properly-educated teachers, using a proper selection of Project materials, can achieve any one of the goals listed above. Each is known to be possible because each has been accomplished.

It is not a matter for elaborate "science" to "prove" such a possibility.

Each has been done, and the order of magnitude of the outcomes have been large enough that one no more needs subtle measures in this case than one does to determine that a house has burned to the ground. The data are unmistakably evident.

Before considering some of the studies of the effectiveness of Project materials, it is worth opposing the Project's own position against the "conventional wisdom." The Project would argue that, where "evaluation" is in fact a matter of *values*, it is not "objective" (in the sense of not involving values), and it is not in fact a matter of common agreement. Parents, teachers, administrators, critics of the schools, and children themselves do not agree, when they have bona fide choices presented to them as alternatives. Furthermore, rather than gathering all possible data -- much of which will merely increase the already excessive homeostatic effect of education -- one should gather only such data as is actually relevant to a possible decision, and only when the cost of obtaining the data does not exceed its value. (It should also be realized that the "costs" of collecting data are much more than merely matters of money; they include effects on schedules, morale, perceptions of autonomy, interference with the integrity of curriculum planning, and other important matters.)

We now look at some of the kinds of data that have been collected and used in various ways. *The main hope is that various investigators will be moved to study some of these matters in greater detail in the future.*

(One question that deserves more thought than it has received is *who* has *what kinds* of responsibility for studying the effectiveness of innovations. It would seem that reports by the innovators themselves are subject to possible conflicts of interest. On the other hand, reports by "disinterested" parties have many weaknesses, including the fact that such parties usually are *not* actually disinterested, that they may not understand the goals of the innovation or the proper methods of employing it, and that they are not usually able to utilize the innovation, but only their own adaptation of the innovation. When one considers how difficult it is to settle such questions as the safety value of automobile seat belts, the danger of cigarette smoking, or the harmfulness of marijuana -- each of which is far simpler than the subtle matters of education -- it would be exceedingly naive to imagine that there can exist a simple empirical method for selecting educational curricula or instructional procedures in a truly wise fashion. Certainly, *if genuine alternatives can be made available*, it will be a matter of protracted controversy and agonizing re-appraisals that will inevitably be involved in making a genuine choice.

Moreover, there is no reason to believe that all parents will make the same choice. Indeed, there is every reason to doubt it. It must further be realized that education is probably far more a matter of *inspiration* and the imparting of values than it is of "achievement levels" in narrowly-defined content areas.)

1. The most formal study made was conducted for the Project by J. Robert Cleary of Educational Testing Services. We reproduce the Cleary report here:[97]

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[97] It should be noted that the curriculum involved in the Cleary study was the sophisticated (but unstable) curriculum designated as "Curriculum ϵ." Cf. Appendix G, and the main body of the report.

A STUDY OF TEST PERFORMANCE IN TWO MADISON PROJECT SCHOOLS AND ONE CONTROL SCHOOL

Presented as a Report to the Director of the Madison Project

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Evanston, Illinois
(Consultant to the Project)

Spring 1965
Data Collected Summer 1964

BACKGROUND

From the beginning of the Madison Project, there have been discussions about evaluation. In one sense, all of the elements which make the Madison Project unique (pre-service teacher training, in-service training, demonstration teaching, the logistics of pilot schools, and materials preparation) were evaluated by the Director and Project staff as they were developed -- the constant revision and critical appraisal of kindescopes and films, and the panel to review teachers' taped lessons, to mention only two.

Viewed within the scope of the entire project, it is not surprising that measurement of student behavior was a later consideration. It required that mathematical experiences related to the Project be provided and that they exist as a vital part of the program of several schools, before any effort at measurement of student behavior could take place.

As these programs began to take shape and as we discussed beginning plans for measurement, uppermost in our minds was the principle that whatever we did in this phase of Project activity should be done consistent with the spirit and broad purpose of the Madison Project.

This is easier to say than to accomplish. In addition to the usual problems which face those measuring new curricula, special features of the Project introduced special problems. Even the usual ones took on special "twists" as we viewed them in the light of Project spirit and purpose.

As further background information, the following statements are presented in an effort to describe the problems as viewed by the Director and this consultant. These concerns also suggest guidelines as we begin to contend with problems in this phase of Project activity:

(1) The purpose of measurement is not to assign value to the Project in relation to something else. Since the Project is a one-day-a-week program, it is not designed to compete with or supplant other programs.

(2) The traditional distinction between measurement and evaluation is even more necessary than usual in beginning this phase of activity. Because of #1 above and other considerations, "evaluation of" the Madison Project is inappropriate.

(3) If "evaluation of" the Project is inappropriate at this stage, so also is the term "evaluation in" the Project for although the programs now seem to be "going" ones in several of the schools, adequate means for assessing all of the important dimensions of the Project have not been fully developed. It is doubtful that we will ever find suitable techniques for some of these dimensions.

(4) Unlike some other programs, content in the Project is viewed as arbitrary. "Content," however, must be used as a vehicle even when "process" is the focus of measurement.

(5) Due to considerations in #4 and others related to methodology of the Project, we have approached activity in this phase with ambivalence. Content has a way of "upstaging" the reason for it. To those who are close to the Project, content is instrumental, but for others it may soon be treated in as pedantic a manner as any other. In particular, we have been concerned that the paper and pencil test, if and when it is used, not work at cross purposes with the Project by suggesting to anyone that what is covered in the test is the Madison Project.

(6) The paper and pencil test cannot be representative of what the Project is for at least two reasons: First, any set of items appearing in the test is only a sample of an almost unlimited number of items from which these were drawn; second, and more importantly, the paper and pencil test is helpful in measuring only a small segment of Project concern. By far the largest segment and most dimensions remain stubbornly resistant to objective measurement. Other techniques will be necessary.

PURPOSE AND RATIONALE

The previous section of this report is far too elaborate a one for the report of this very small pilot project in measurement. However, it was considered necessary to place it in proper prospective as well as to take this opportunity to state more formally the guidelines we have developed in discussion over time.

In light of the previous section, then, it was decided that this first, small, formal attempt at measurement be entirely a problem of "description" not "evaluation."

More specifically, it would be a small scale effort to describe how Madison Project students operated on fairly difficult material of both traditional and more contemporary content but with the more traditional notation. By using content from other sources expressed in notation different in degree from that used in the Madison Project, we avoided the problem discussed in #5 and #6 above, and at the time we thought we could gain some insight into the side effects of the Project on more traditional tasks.

The investigator views this pilot study as

(1) More illustrative and suggestive than definitive

(2) More for the purpose of planning the future than for documenting the past

(3) More for clarification than for enlightenment.

PROCEDURE

With the kind cooperation of Dr. Roger Lennon, Director of the Test Division of Harcourt, Brace and World, Inc., this investigation was able to obtain the pretest statistics on approximately 400 items from the pre-publication forms of the new mathematics test of the Stanford Achievement High School Test Battery.

Forty-five items of this array were chosen by attention to their relevance to the Madison Project and the item statistics from the pretest. A calculated guess was made that seventh grade Madison Project students would perform as well as a sample of ninth grade students taking some variety of modern math in schools similar to Madison Project pilot schools chosen for this study, and items were chosen from this sample of pretest data supplied by Harcourt, Brace and World, Inc., so as to yield an average item difficulty of 50-60%.

Two Project schools, Weston, Connecticut and Clayton, Missouri, and a third school in Wilton, Connecticut, similar to the Project schools but not using the Madison Project, participated in this study. One class of seventh grade students in each of these schools were tested. All three schools were judged to have approximately equal characteristics, such as socio-economic background, parent educational and professional level, staff and facilities, and student ability. (Another study, not reported here, established that there were no significant differences in intelligence test scores.)

Items were assembled by topic and organized into a test with a separate answer sheet provided and administered in each of the participating schools by Project staff members within a three-week period in the spring of 1964. Test scoring and the following test analysis was completed by this consultant.

ANALYSIS

For the purpose of presentation of the results of this report, an attempt has been made to organize the data and interpretation into parts concerned with the test, the schools, and the items. Some overlap in presentation, however, is unavoidable.

The Test Analysis

Ample working time was given by the examiners to insure that the test would be a work limit measure rather than a speeded test. An hour was provided for administration and working time. 87% of the students completed the entire test, although omitting responses on difficult items rather than guessing these answers.

Table 1, Page A-11, reports the distributions of raw scores for the three schools. Table 2, Page A-12, presents the combined distributions. Referring to these two tables indicates that our estimates of the difficulty of the items for this group was correct. The mean score of the combined distribution is 25.53. This mean is 57% of the total raw score. Inspection of the raw score distributions and reference to the standard deviation, demonstrate that the dispersion of the group is excellent affording nearly optimum discriminating power for the group tested.

In addition, no student approached a perfect score and only 3 students of the 77 tested show no evidence of recording knowledge on the test -- less than 4% of the group (this figure resulted from using the formula

$S_c = \left(\frac{1}{A}\right)\sqrt{K\ (A-1)}$ and the chance score as $\frac{K}{A}$, where

S_c = Standard deviation of the chance scores

TABLE 1
DISTRIBUTIONS

RAW SCORE	SCHOOL A	SCHOOL B	SCHOOL C
42–45			
41	1		
40		1	
39			
38		1	
37	1		
36			
35	1	1	
34	1	1	
33			1
32	3		
31	1		
30	1	2	1
29		3	1
28	1	1	2
27	3	1	2
26	1	1	2
25	1	2	3
24	3	3	3
23	2	1	1
22			5
21	2	1	2
20	1		1
19	2		2
18		3	
17			1
16			
15		1	
14			
13		1	
12			
11			
10		1	

TABLE 2
COMBINED DISTRIBUTION

Schools A, B, C

RAW SCORE	f	X	X^2
41	1	41	1681
40	1	40	1600
39			
38	1	38	1444
37	1	37	1369
36			
35	2	70	2450
34	2	68	2312
33	1	33	1089
32	3	96	3072
31	1	31	961
30	4	120	3600
29	4	116	3364
28	4	112	3136
27	6	162	4374
26	4	104	2704
25	6	150	3750
24	9	216	5274
23	4	92	2116
22	5	110	2420
21	5	105	2205
20	2	40	800
19	4	76	1444
18	3	54	972
17	1	17	289
16			
15	1	15	225
14			
13	1	13	169
12			
11			
10	1	10	100

N = 77 Students

$\overline{X}$ (Mean Raw Score) = 25, 53

S (Standard Deviation Raw Score Units) = 5.96

A = The number of options to each item

K = The number of items in the test (the perfect score)

and where a score greater than $\frac{K}{A} + 2\ S_c$ is used as a minimum score necessary to demonstrate knowledge.*)
Reliability is considered later.

The Schools

Table 3, below, presents means, standard deviations, and t values for the distributions of the three schools.

TABLE 3
MEANS, STANDARD DEVIATIONS, AND t VALUES OF STUDY SCHOOLS

	SCHOOL A	SCHOOL B	SCHOOL C
N =	25	25	27
$\overline{X}$ (Raw score)	27.28	25.32	24.11
S (Standard deviation)	5.73	7.28	3.64
	$\overline{X}_A = \overline{X}_B$	$\overline{X}_A = \overline{X}_C$	$\overline{X}_B = \overline{X}_C$
t Value	1.89	2.29*	.74

* Significant at .05 level

The only test planned in advance was that to test the null hypothesis that

$$\overline{X}_A = \overline{X}_B = \overline{X}_C$$

(that the observed differences were those likely to have occurred by chance). If this were not true, then we might infer that any difference might be a product of different conditions or programs of the schools.

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* Gulliksen, Harold, Theory of Mental Tests, Chap. 19, p. 304, 1950, John Wiley and Sons, Inc., New York.

The only significant difference when the probability Type I error is .05 occurs when Schools A and C are considered. The fact that none occurred between Schools B and C was somewhat surprising. Therefore, the consultant spent two days in the school which had not used the Madison Project to learn more about the students tested and the school program.

A sample of 15 students tested were interviewed individually and asked to verbalize their attack on selected items from the test. These interviews were tape-recorded and later printed in full for analysis. In addition, a content analysis was done on materials of instruction, and an interview with the teacher was conducted. Although the results of these activities will not be covered in detail in this report, enough evidence was gathered to explain the rather good test performance of these students. This explanation will be given in the final section, Conclusions and Implications, of this report, and some of this analysis will be presented in this section under The Items.

To conclude this section, there is no evidence to indicate significant differences between School A and School B nor between School B and School C. The only possibility of difference in mean performance on the test occurs between School A and School C. Since most of the content of the test is not specific to the Madison Project, these results are expected; still, they were intriguing to the consultant and were therefore investigated more fully.

The Items

Table 4, Page A-16, presents the item response by school. Due to the small N in this study, option comparisons with publisher's pretest data are not given, because the publisher traditionally presents data by the upper and lower 27% of those tested. Item difficulty indices of Schools A, B, and C combined are, however, compared with the item difficulty indices from the total group of the publisher's pretest data in Table 5, Page A-20. Note in Table 5 that on most items, 7th graders in the study schools exceeded the performance of 9th graders in the publisher's pretest group.

Table 6, Page A-22, presents the item difficulties by school and also indicates differences in relative difficulties of each item, when the schools are compared in pairs. The Sign Test for Paired Observations* was used to test for significant differences both between the study data and the pretest data and the pretest sample in Table 5 and between the study schools themselves in Table 6. The schools in the study perform significantly higher than those in the publisher's sample ($P =$ less than .01). There are no significant differences between the schools of the study at point $P = .05$ or less.

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* Edwards, Allen L., Statistical Methods for the Behavioral Sciences, p. 288-289, Rinehart and Co., Inc., New York, 1954.

Approximately 9-10 items appear too easy for the study schools and another 6 are either too hard or need revision because of inadequate discriminating power. The easy items are those related to linear equations and substitution therein, prime numbers, and square root; the difficult or ambiguous items are related to the distributive law, matrix algebra, and the analytic geometry type item calling for the ability to relate a function other than linear to its graph.

An interesting pattern results from an inspection of the right-hand portion of Table 6 which presents the signs of the school comparisons. School C is the school not offering the Madison Project. Students in School C equal or surpass performance in the other schools on two blocks of items only -- the first 9 items and items 17-22, except item 19 (algebraic notation). Except these items and about 6 other miscellaneous items through the remainder of the test, the sign of the School C distribution in the other items is "minus" indicating a lower percent passing.

Five of the first 9 items measure the arithmetic of signed numbers; the last 4 items in this group are of quadratic equation form calling for the roots (solution set). Before the taped interviews with students in School C, mentioned earlier, performance on these quadratic equation items was surprising, since this topic has been in Madison Project classes as early as the third and fourth grades. The taped interviews disclosed, however, that students in School C did not solve these equations as intended. Rather they substituted the roots or numbers of the solution set until they found the correct pair. This was documented on the tapes both by conversations and by noting the typical errors on solution attempts. The most common first step with School C youngsters was to transpose the equation so that the X^2 term remained alone left of the = sign. Further documentation of the lack of knowledge of quadratics was gained by arranging for this consultant to teach these students for one period. This further established the fact that they did not know how to attack quadratic equations. This is understandable also from the content analysis performed on the instructional materials and by the interview with the teacher.

TABLE 4
ITEM RESPONSES BY SCHOOL

ITEM NO.	SCHOOL A	SCHOOL B	SCHOOL C
1	a 0 c 0 b 0 d 25	a 0 c 1 b 1 d 23	a 0 c 0 b 0 d 27
2	a 4 c 2 b 16 d 3	a 7 c 1 b 9 d 8	a 2 c 7 b 17 d 1
3	a 3 c 0 b 0 d 22	a 7 c 0 b 0 d 18	a 2 c 0 b 0 d 25
4	a 0 c 1 b 0 d 24	a 2 c 9 b 0 d 14	a 0 c 1 b 0 d 26
5	a 20 c 0 b 4 d 1	a 14 c 1 o 1 b 5 d 4	a 25 c 0 b 1 d 1
6	a 6 c 2 b 4 d 13	a 1 c 3 o 2 b 6 d 13	a 1 c 2 b 7 d 17
7	a 13 c 3 o 2 b 4 d 3	a 13 c 5 o 2 b 1 d 4	a 15 c 2 o 1 b 4 d 5
8	a 7 c 3 o 2 b 4 d 9	a 1 c 6 o 2 b 9 d 7	a 3 c 7 o 1 b 1 d 15
9	a 2 c 6 o 1 b 13 d 3	a 5 c 12 o 2 b 6 d 0	a 1 c 5 b 15 d 6
10	a 3 c 16 b 4 d 2	a 2 c 14 b 6 d 3	a 10 c 9 b 8 d 0
11	a 1 c 14 b 4 d 6	a 3 c 12 b 4 d 6	a 9 c 11 o 1 b 0 d 6
12	a 4 c 2 b 2 d 17	a 4 c 1 o 1 b 3 d 16	a 11 c 1 o 1 b 1 d 13

TABLE 4 (Continued)

ITEM NO.	SCHOOL A	SCHOOL B	SCHOOL C
13	a 2 c 1 b 18 d 4	a 5 c 0 b 12 d 8	a 5 c 2 o 1 b 19 d 0
14	a 10 c 5 o 3 b 7 d 0	a 12 c 2 o 2 b 9 d 0	a 3 c 2 o 7 b 9 d 6
15	a 1 c 14 o 2 b 7 d 1	a 3 c 16 o 2 b 4 d 0	a 0 c 12 o 6 b 6 d 3
16	a 1 c 5 o 2 b 15 d 2	a 0 c 12 o 2 b 5 d 6	a 4 c 4 o 9 b 8 d 2
17	a 25 c 0 b 0 d 0	a 21 c 2 o 2 b 0 d 0	a 27 c 0 b 0 d 0
18	a 0 c 23 b 0 d 2	a 0 c 16 o 4 b 0 d 5	a 0 c 24 b 0 d 3
19	a 14 c 5 o 2 b 2 d 2	a 10 c 8 o 2 b 1 d 4	a 10 c 5 o 2 b 5 d 5
20	a 0 c 1 b 0 d 24	a 0 c 4 o 1 b 1 d 19	a 0 c 0 b 1 d 26
21	a 1 c 1 b 0 d 23	a 2 c 0 o 2 b 0 d 21	a 1 c 0 b 0 d 26
22	a 1 c 1 b 0 d 23	a 0 c 1 o 2 b 1 d 21	a 0 c 0 b 0 d 27
23	a 6 c 9 o 1 b 4 d 5	a 1 c 12 o 2 b 5 d 5	a 4 c 10 o 4 b 4 d 5
24	a 5 c 2 b 3 d 15	a 2 c 5 o 3 b 1 d 14	a 3 c 1 o 1 b 1 d 21
25	a 4 c 17 b 1 d 3	a 1 c 17 o 1 b 4 d 2	a 1 c 16 o 6 b 0 d 4

TABLE 4 (Continued)

ITEM NO.	SCHOOL A	SCHOOL B	SCHOOL C
26	a 1 c 3 **b 21** d 0	a 0 c 3 **b 22** d 0	a 0 c 2 o 1 **b 24** d 0
27	a 5 c 7 **b 12** d 1	a 4 c 5 **b 14** d 2	a 12 c 3 **b 8** d 4
28	a 6 c 2 b 1 **d 16**	a 2 c 3 o 2 b 3 **d 15**	a 6 c 7 o 3 b 4 **d 7**
29	**a 25** c 0 b 0 d 0	**a 24** c 1 b 0 d 0	**a 17** c 6 o 1 b 3 d 0
30	**a 24** c 0 b 0 d 1	**a 24** c 0 o 1 b 0 d 0	**a 24** c 0 o 1 b 2 d 0
31	**a 17** c 3 o 1 b 0 d 4	**a 17** c 1 o 2 b 2 d 3	**a 16** c 3 o 2 b 2 d 4
32	**a 15** c 3 o 1 b 5 d 1	**a 16** c 1 o 3 b 1 d 4	**a 6** c 4 o 8 b 4 d 5
33	a 3 **c 16** b 0 d 6	a 9 **c 9** o 2 b 3 d 2	a 6 **c 3** o 6 b 6 d 6
34	a 2 **c 11** o 2 b 8 d 2	a 2 **c 12** o 3 b 5 d 3	a 2 **c 8** o 7 b 8 d 2
35	a 1 **c 22** b 1 d 1	a 0 **c 21** o 1 b 2 d 1	a 1 **c 20** o 3 b 1 d 2
36	a 5 c 9 o 1 **b 5** d 5	a 4 c 12 o 1 **b 5** d 3	a 0 c 13 **b 7** d 7
37	a 5 c 2 o 4 b 8 **d 6**	a 2 c 3 o 4 b 10 **d 6**	a 4 c 5 o 8 b 6 **d 4**
38	a 7 **c 6** o 2 b 6 d 4	a 7 **c 3** o 4 b 5 d 6	a 6 **c 1** o 8 b 10 d 2

TABLE 4 (Continued)

ITEM NO.	SCHOOL A	SCHOOL B	SCHOOL C
39	a 0 c 2 o 1 b 18 d 4	a 2 c 4 o 1 b 15 d 3	a 2 c 5 o 3 b 13 d 4
40	a 0 c 1 o 1 b 0 d 23	a 2 c 0 b 0 d 23	a 1 c 2 o 1 b 0 d 23
41	a 11 c 1 o 2 b 4 d 7	a 11 c 5 b 5 d 4	a 13 c 3 o 6 b 4 d 1
42	a 5 c 4 o 4 b 8 d 4	a 4 c 4 b 15 d 2	a 7 c 3 o 2 b 10 d 5
43	a 1 c 8 o 4 b 10 d 2	a 1 c 11 o 1 b 8 d 4	a 2 c 12 o 6 b 6 d 1
44	a 4 c 14 o 5 b 1 d 1	a 1 c 13 o 6 b 3 d 2	a 4 c 7 o 10 b 3 d 3
45	a 1 c 0 o 6 b 13 d 5	a 3 c 1 o 1 b 11 d 9	a 2 c 1 o 3 b 9 d 12

TABLE 5

ITEM DIFFICULTIES -- (% SUCCEEDING)

ITEM NO.	TOTAL SCHOOLS A,B,C N=77, 7th GRADE	PUBLISHER'S PRETEST GROUP 9th Grade
1	.97	.86
2	.55	.53
3	.84	.69
4	.83	.75
5	.77	.68
6	.56	.20
7	.53	.32
8	.40	.21
9	.44	.36
10	.51	.59
11	.48	.60
12	.60	.63
13	.16	.31
14	.32	.15
15	.48	.27
16	.36	.15
17	.95	.93
18	.82	.63
19	.44	.35
20	.90	.82
21	.91	.87
22	.92	.94
23	.40	.48
24	.65	.51
25	.65	.60
26	.87	.71
27	.44	.51
28	.49	.43
29	.86	.74
30	.93	.91
31	.65	.38
32	.48	.42
33	.36	.29

TABLE 5 (Continued)

ITEM DIFFICULTIES -- (% SUCCEEDING)

ITEM NO.	TOTAL SCHOOLS A,B,C N=77, 7th GRADE	PUBLISHER'S PRETEST GROUP 9th GRADE
34	.40	.21
35	.82	.76
36	.22	.21
37	.21	.25
38	.13	.13
39	.60	.55
40	.90	.92
41	.45	.45
42	.43	.64
43	.09	.10
44	.44	.39
45	.34	.20

TABLE 6

ITEM DIFFICULTIES BY SCHOOL -- COMPARED IN PAIRS BY THE SIGN TEST

	(DIFFICULTY %)					
Item no.	School A	School B	School C	B compared to A	C compared to A	C compared to B
1	100	92	100	-	0	+
2	64	36	63	-	-	+
3	88	72	94	-	+	+
4	96	56	96	-	0	+
5	80	56	94	-	+	+
6	52	52	63	0	+	+
7	52	52	56	0	+	+
8	36	28	56	-	+	+
9	52	24	56	-	+	+
10	64	56	33	-	-	-
11	56	48	41	-	-	-
12	68	64	48	-	-	-
13	16	32	0	+	-	-
14	28	36	33	+	+	-
15	56	64	44	+	-	-
16	20	48	15	+	-	-
17	100	84	100	-	0	+
18	92	64	89	-	-	+
19	56	40	37	-	-	-
20	96	76	96	-	0	+
21	92	84	96	-	+	+
22	92	84	100	-	+	+
23	36	48	37	+	+	-
24	60	56	78	-	+	+
25	68	68	59	0	-	-
26	84	88	89	+	+	+
27	48	56	30	+	-	-
28	64	60	26	-	-	-
29	100	96	63	-	-	-
30	96	96	89	0	-	-
31	68	68	59	0	-	-
32	60	64	22	+	-	-
33	64	36	11	-	-	-
34	44	48	30	+	-	-
35	88	84	74	-	-	-

TABLE 6 (Continued)

	DIFFICULTY (%)					
Item no.	School A	School B	School C	B compared to A	C compared to A	C compared to B
36	20	20	26	0	+	+
37	24	24	15	0	-	-
38	24	12	4	-	-	-
39	72	60	48	-	-	-
40	92	92	85	0	-	-
41	44	44	48	0	+	+
42	32	60	37	+	+	-
43	8	16	4	+	-	-
44	56	52	26	-	-	-
45	20	36	44	+	+	+

The substitution knowledge of School C youngsters explains a great deal. Nearly every item on which School C youngsters performed better than Madison Project students was either an arithmetic item or an item requiring formula substitution skill (both in linear and quadratic equations). On all other items dealing with algebraic knowledge, graphic interpretation of functions linear and non-linear, and other mathematical topics (more central to some of the aspects of the Madison Project), School C did less well.

The only discernible pattern between the two Project schools, School A and School B, occurs in items 14-16, where the performance of School B surpasses that of School A as a block of items dealing with matrix algebra. Knowing something about School B and the experiences provided would support the hypothesis that School B had had more experience in matrix algebra than School A. Information was not gathered to support this hypothesis. On only 9 more items in various parts of the test did School B out-perform School A, still the difference in mean performance was not significant.

Test Reliability

Kuder-Richardson Formula 20 was used to obtain an estimate of the reliability of the test for the study schools. The reliability thus obtained was .78 which is an underestimate of the reliability. Considering the small N, similarity of the groups, and their high level this is a very satisfactory index.

Conclusions and Implications

The data from this pilot study suggest the following conclusions:

(1) The test prepared from the larger array of items is a good one technically. Revision of some items are necessary, if the test were to be used again for similar groups.

(2) A good deal of useful and valid information about the outcomes of instruction can be obtained from such an instrument.

(3) Madison Project students perform verywell on more traditional content and at a higher level than 9th grade students in similar schools having some variety of "modern" mathematics.

(4) Statistics such as individual and group total scores, and mean performance disguise more than they disclose, when programs of study have been different in content and exposure to this content.

Three implications of this study are viewed as important by this consultant:

(1) Items dealing with quadratic equations, in particular, need revision, in order to prevent substitution skill per se to be sufficient for solution. (Items dealing with simultaneous linear equations may also need revision.)

(2) In the relatively minor role which objective measurement can play in "description" of the Project, we should be encouraged that carefully controlled measurement will yield satisfactory results and contribute satisfactory information about what students can do as a result of learning experiences, but more importantly

(3) If objective measurement is used as a part of such description, it must be accompanied by much more information about the learning environments in which this measurement took place.

Particularly in the area of school mathematics which has many variations in the form of new curricula at all levels, specific information about environments must be gathered and used to interpret results of measurement. Without such analysis, results are highly suspect, if not misleading.

A set of recommendations to investigate such analysis of the learning environment will be presented later to the Director.

It seems likely that quite subtle, diverse, and extensive methods must be employed in any serious attempt to describe the effects of variations in curriculum and instruction in mathematics. Data reported by Rosenthal and others indicate that personality characteristics, sylistic traits, and sex of teachers interact in a complicated way with the personality, style, and sex of each child, and that these interactions have an important relation to goals, content, and methods of instruction. The forthcoming S.M.S.G. "longitudinal study" may represent the first serious attempt to take account of many variables of known importance. The fact unfortunately remains that variations in curriculum and instruction from one school to another are at present still exceedingly slight. This can therefore hardly be used as a genuine "independent variable."

Notice that, in the Cleary report, the three 7th grade classes tested performed better than national norms for grade 9 for good schools. At the very least, a two-year head start on high school and college mathematics is possible in grades K-7, for above-average achievers.

Another factor, not mentioned in the Cleary report, but observed at that time, is the seemingly great importance of group dynamics in different classrooms. Some groups of children work together to mutual advantage; in other classrooms rivalry, status fights, etc., extract a heavy toll both in morale and in content achievement. (One interesting study on this is reported in: Paul De Hart Hurd and Mary Budd Rowe, "A Study of Small Group Dynamics and Productivity in the BSCS Laboratory Block Program," (52).

2. A second, less formal study of the use made of Project materials in the hands of many different teachers was conducted by the method of tape recording actual classroom lessons by these teachers, duplicating the tapes, and sending a copy of each tape to each of sixty professional panelists. Each panelist prepared a report on each lesson, giving his own analysis and interpretation. These sixty reports were then analyzed, and a composite report was prepared. Panelists were not told in advance what to observe, but focussed on whatever aspects caught their attention. The detailed reports are available from The Madison Project, 8356 Big Bend Blvd., Webster Groves, Missouri 63119. One result appeared to stand out: many observers identified some teachers as primarily concerned with "the way things ought to be," "the way you ought to solve such-and-such a problem," "the way a teacher (or a child) ought to behave," and so on. They had some concept of correct behavior, and judged themselves and children according to this.

By contrast, another identifiable group of teachers seemed to relate their goals and methods more closely with the children, to be more aware of what the children liked, disliked, tended to do well, etc. A psychoanalytically-oriented psychiatrist on the panel identified these groups as exhibiting, in the first case, "super ego-dominated ego function," and in the second, "id-dominated ego function." A teacher, referring to the first group, wrote: "More orders from the Giant People!" It has been the consistent opinion of most observers, both on the panel and since then, that the "ought" people are generally less successful in teaching Madison Project materials. This, also, may deserve further study, but a curriculum project cannot allow itself to pursue such matters as far as they probably deserve, without risking a loss of momentum in the basic curriculum effort itself. (To confound matters further, there is some indication that rather rigid, compulsive teachers, if they can allow themselves to relax somewhat, can teach Project materials very well indeed, especially in dealing with children who tend to misbehave.)

3. The third study conducted by the Project was the following:

Tape-recorded interviews of individual 6th and 7th grade children, conducted by Herbert Barrett.

The Project had noticed that children in many school programs, both "traditional" and otherwise (and including many students in Madison Project classes, also), showed a deterioration in their performance from grade 5 through grade 7. They were at their best in grade 5 or in early 6th grade, and did less well thereafter.

In an attempt to gain some comprehension of this phenomenon -- which the Project does not claim to understand even now -- a clinical psychologist spent a year interviewing individual children from grades 6 and 7, and tape-recording these interviews. He thereafter studied these tapes and reported on what the children seemed to be saying about their experiences in school, their goals, their problems, etc. The children were not aware that the study was concerned with mathematics.

The most striking result, which emerged rather clearly, was that the children liked those subjects which involved physical activity and opportunities to talk to other children, and disliked those subjects which involved sitting still, and which offered no opportunities to talk with friends. There is no interpretation here; the children were quite articulate and quite explicit. They disparaged subjects where "all you do is sit and read and write." They liked orchestra, chorus, physical education, laboratory work, and art work; they disliked Latin, modern languages, social studies, English, and mathematics.

It was apparent that they were "original" and "clever" in those subjects which they reported they liked, and were neither very original nor very clever in those subjects they disliked.

(This phenomenon may deserve further study; it is complicated, since it not only varies from school to school and from one region of the United States to another, but it is not particularly apparent as long as only mediocre tasks are involved; it becomes exceedingly noticeable when one asks the children to perform much more sophisticated tasks, such as to make up their own set of axioms to describe various algebraic structures, and to use these axioms to prove various theorems. Bright *fifth-graders* appear to do this better than bright *seventh-graders*!)

Perhaps the most important fact is that the data collected in the Herbert Barrett study *did*, indeed, form the basis for a decision which has been implemented: the Project has moved further away from an exclusively paper-and-pencil approach and has come to make extensive use of physical materials and "mathematics laboratories" *at all grade levels*, K-9, *and also in college courses for prospective teachers.* The response of both school and college students has strongly confirmed the appropriateness of this direction, and so has the fact that many other projects and individuals have moved, independently, in this same direction. Foremost among these is the English Nuffield Mathematics Project, directed by Geoffrey Matthews. An excellent presentation of English work is given in the film *I Do ... And I Understand,* available from Mr. S. Titheradge, Manager, New Print Department, Sound Services, Ltd., Wilton Crescent, Merton Park, London, S.W. 19, England. Other individuals prominent in this movement have been Leonard Sealey, Lauren Woodby, Edith Biggs, Marion Walter, Margeurite Brydegaard, Mary Lela Sherburne, Z.P. Dienes, Paul Merrick, Emily Richard, William Fitzgerald, Caleb Gattegno, John Trivett, Jerrold Zacharias, Donald Cohen, Gerald Glynn, Victor Wagner, and David Hawkins. (Attention should also be paid to the English Association of Teachers of Mathematics.)

There are also a number of quite interesting studies in the use of Madison

Project materials that have been performed independently of the Project. Before considering these, it is worth taking a further brief look at what kinds of data can be collected for what kinds of purposes, and at some of the assumptions that are usually involved.

i) Assumptions of Repeatability: The conventional wisdom tends to be optimistic about the prospects for repeatability. The Project is far less so; the world is changing, children are changing, schools are changing, parental expectations are changing, teacher education programs are changing, available textbooks and visual aids are changing, college admission influences are changing, budgets are changing, the entire social milieu is changing -- consequently, "curriculum and instruction" must be viewed in a historical sense. (Notice that nearly every science is faced with the need to incorporate a more historical approach; for example, medical science is faced with bacteria which quickly develop strains that are resistant to various antibiotics, and astronomy is becoming increasingly concerned with the historical evolution of the universe, In some cases, as these two examples show, historical evolution proceeds very rapidly, in others very slowly -- but it does proceed.)

In curriculum work the situation changes enough in two or three years so that literal repeatability of total experiences may be impossible or irrelevant.

(Notice that television has made scales of reading difficulty obsolete, and that new brand names sometimes alter psychologists' lists of nonsense syllables.)

ii) However, the Project's practice of recording actual lessons on film greatly increases the repeatability of the present program.

iii) Assumptions of "objectivity." The conventional wisdom attaches considerable virtue to "objectivity," but is both vague and inconsistent in the meanings it attaches to the word "objectivity." Sometimes this word appears to refer to matters which are conducted openly in the public view (as when decision procedures are carefully specified and the specifications are carefully followed), sometimes it appears to refer to something in which human values (and consequent disagreements) are not involved, and sometimes it seems to refer to something where questionnaires have been used or numerical data collected. (There are many other common meanings.)

The Project's position is that specific methodology is at most incidental, that the non-involvement of values is impossible, but that openness to public forums and discussions is in fact essential to progress. The Project has indeed conducted all of its work as openly as possible, through films showing actual classroom lessons, through visitors observing demonstration centers, and through presentations and discussions at professional meetings, conferences, symposia, etc.

iv) The Collection and Use of Feedback Data.

On a relatively microscopic level, the Project obtains vast amounts of feedback data, via a maximum number of channels, because Project teaching is based on a high degree of student involvement, because of the Project's emphasis on a "process" approach, because of the use of small-group work in classes, and because lessons are designed and revised right in the classroom, with the participation of the students.

Further steps in testing the Project's prototype lessons have been discussed in the body of this report, including "viability" or "stability" testing -- that is to say, trying out a lesson sequence in a variety of settings and with a variety of teachers, in order to see if the sequence works more or less reliably under a variety of conditions.

In addition, the Project staff have engaged in very extended discussions and argumentation concerning the teaching of various lessons or of specific content topics. Somehow rational discussion has come to be lightly regarded by many students of education, but this is conspicuously unjustified; rational discussion (augmented by a capability of making proofs) is the single foundation of mathematics itself, which nowadays largely rejects any empirical foundations for most of its efforts, and rational discussion is surely the main way in which we cope with nearly every decision in nearly every line of human endeavor. To imagine that quantitative methods will gradually eliminate this is almost certainly foolish: the quantitative methods themselves are selected by, applied by, and based upon rational discussion. The task of conceptualizing reality -- however much use it may make of whatever devices -- must have its foundation in rational discussion. Writing in Perspectives on Education, Dean Robert J. Schaefer recently discussed the "search for modes of criticism of the art of teaching," and noted:

> There are many modes of analysis which can be

> applied to the classroom. ... On several occasions John Fischer has suggested the development of a criticism of teaching comparable to established traditions in literature and the arts -- an aesthetics of the art of teaching, so to speak.[98]

This "aesthetics of the art of teaching" -- or, in Polanyi's terms, this collection of "practitioner's maxims" -- has been agressively pursued in protracted discussions among Project staff over a period of more than eight years. In one or two cases, thanks to 16 mm. film, the exact same lesson has been viewed by the same observer as many as several hundred times, without having exhausted the possibility of new insights even at the end. These discussions have done far more to shape the Project's prototype lessons -- as well as its "theory of instruction" -- than any systematic quantitative studies have been able to do. In the Project's opinion, a three thousand dollar film of a lesson will usually contain more information, and more useful information, than a three thousand dollar quantitative study of teaching will.

It should be recalled that in developing prototype lessons, the Project had expressly chosen not to aim primarily at establishing "scientific generalizations," since this did not seem, in the 1960's, to be the most effective path to significant curriculum improvement. As late as 1967 there still seems no reason to question this decision.

The preceding remarks are intended to indicate that the Project has not been contemptuous of, or negligent of, attempts to comprehend reality and to cope with it in humanly useable terms, even if its procedures have frequently been at variance with the conventional wisdom concerning the study of education.

We turn now to various studies, both formal and informal, that have dealt with Madison Project materials but have been undertaken by others, quite independently of the Project itself. The Project is always grateful for such studies, and hopes that their number will increase.

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[98] From: The School as a Center of Inquiry by Robert J. Schaefer. Reprinted by permission of the publisher, Harper and Row. Examples of Fischer's remarks can be found in (38), and in the Annual Report Issue of Teachers College Bulletin, 1963-1964, pp. 1-9.

4. The most formal of these is probably the doctoral thesis by Charles D. Hopkins, entitled *An Experiment on Use of Arithmetic Time in the Fifth Grade*, Indiana University, June, 1965.

This study was concerned with what happens to a child's proficiency in "traditional arithmetic" if some of the time ordinarily devoted to "traditional arithmetic" is diverted to the study of such other mathematical topics as Cartesian coordinates, open sentences, graphs, and the arithmetic of signed numbers (cf. Hopkins, op. cit., p. 66). This formal study confirmed a large amount of Madison Project experience (of a less formal sort), namely, that when less time is spent on traditional arithmetic and diverted instead to more interesting mathematics, the students perform better *even on the traditional topics* (which are thus receiving less emphasis). Something of the superior motivation or insight gained from the other topics more than offsets the decrease in time spent on traditional topics.

Notice that Dr. Hopkins kept total time constant for both groups. This, then, is in some ways the most severe test that can be posed concerning the introduction of Madison Project materials. There is, of course, no reason to assume that total time *will* be kept constant in actual practice. Present "after-school arithmetic programs" have lengthened the school day, and some schools have extended the school year to eleven months. Hopkins' results clearly suggest that, if keeping the time constant already favors a reduction of traditional arithmetic and the introduction of more advanced topics, then whenever the total time is actually increased the introduction of more advanced topics should be, *a fortiori*, even more strongly indicated. We do not need as much time for traditional arithmetic as we presently allow; surely, then, we do not need more. The added time should be devoted to more interesting mathematical topics.

Hopkins' fifth grade classes were stratified by ability, and the same conclusion holds for all ability levels within his sample.

It should also be noted that, among all studies performed "outside of the Project," this one probably involved the most authentic use of Madison Project materials. The study was made entirely independently of the Project, and even without the Project's knowledge, but Dr. Hopkins had studied with the Project for several years, and had helped the Project operate the "big city" workshop in Chicago in 1964. The Project has never claimed, and does not claim, that untrained teachers can make effective use of Project ideas. It is for this reason that the Project's efforts at dissemination are directed almost entirely toward teacher education.

5. The doctoral study carried out by Wanda M. Walker as a Ph.D. dissertation at California Western University. For information, contact Mrs. Wanda M. Walker, 13208 Julian Avenue, Lakeside, California 92040.

6. A Master's Thesis by Mariam Y. Cohn, entitled "A Description and Evaluation of a Classroom Teaching Experience Involving the Use of the 'Madison Project' Mathematics Materials," carried out under the direction of Professor George Spooner of Central Connecticut State College, New Britain, Connecticut (August, 1963).

7. Incidental use of Madison Project materials has been made by Patrick Suppes, as a very small part of a very much larger undertaking. Cf. Patrick Suppes, "Accelerated Program in Elementary-School Mathematics -- The Second Year," Psychology in the Schools, Vol. III, No. 4 (October, 1966), pp. 294-307.

This brings us to more informal reports from teachers; while in all of the following cases the materials are recognizably either the Madison Project materials, or else highly similar materials developed independently of the Project, it is in some cases impossible to say what the actual lines of influence -- if any -- may have been. Indeed, this latter fact is a measure of the extent to which small parts of this program, at least, are gaining a relatively wide currency.

8. Robert A. Trimmons, "Tic-tac-toe -- a mathematical game for grades 4 through 9," The Arithmetic Teacher, Vol. 14, No. 6 (October, 1967), pp. 506-508.

This article is of interest for a variety of reasons. It includes the following remarks:

> One of the difficulties with this game is that the

> children would like to play it all the time. I have yet to find a class that tires of it.
>
> I believe the activity originated in the Madison Project. A complete description of it can be found in Discovery in Mathematics: A Text for Teachers, by Robert B. Davis (Reading, Mass.: Addison-Wesley Publishing Company, 1964).
>
> The value of this activity and others like it is the enthusiasm it generates. It allows all to participate no matter what their degree of competency, and lends itself to the discovery method of teaching with little effort on the teacher's part.[99]

This has the authentic flavor, as well as the content, of a Madison Project lesson; students of the diffusion of innovation may be interested to note, however, that the most complete discussion of the Tic-tac-toe game is not given in Discovery in Mathematics, but rather in the book which is a sequel to Discovery, namely:

> Robert B. Davis, Explorations in Mathematics: A Text for Teachers, Addison-Wesley, 1967.

9. A lesson similar to one used by the Madison Project is reported in: C. Winston Smith, Jr., "The intersection of solution sets," The Arithmetic Teacher, Vol. 14, No. 6 (October, 1967), pp. 504-506. Probably this is of separate origin and does not in fact trace its origins to the Madison Project. (Cf. the "Editor's Note" by Charlotte W. Junge on p. 506.)

10. A lesson sequence also possessing the authentic Madison Project flavor, but created entirely independently of the Project, is described in:

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[99] From: Robert A. Timmons, "Tic-tac-toe -- a mathematical game for grades 4 through 9," The Arithmetic Teacher, Vol. 14, No. 6 (October, 1967), pp. 506-508. Reprinted by permission of Marguerite Brydegaard, Editor.

Effie Froelich, "An investigation leading to the Pythagorean property," The Arithmetic Teacher, Vol. 14, No. 6 (October 1967), pp. 500-504.

This is a case where the Project had nothing whatsoever to do with the creation of this lesson sequence, but this is nonetheless precisely the kind of lesson sequence which the Project does seek to create.

Developing creative curriculum materials is not easy. In the same essay cited earlier, Robert Schaefer remarks on "strengthening the capacity for meaningful manipulation of content" as follows:

> Although other pedagogical issues may be involved, dramatic failures in teaching are often due to the discontinuity between content organized for the demands of collegiate specialization and content appropriate to the general education of younger pupils. Translating the patterned abstractions and conventions of upper-level college courses into terms appropriate to an elementary- or secondary-school program requires enormous effort and a considerable intelligence. The difficulties in such translation undoubtedly explain our tendency to shunt academically unresponsive youngsters into shop courses, remedial work in general mathematics, or, in cliche travesty, into Fly-Casting I and II. The difficulties do not excuse us, however.
>
> I suspect, also, that our failure to stimulate interest in the organization of substantive content for instructional purposes explains the disillusionment of many academically able students who elect to teach in the lower schools. Seeking to perpetuate his own intellectual pleasure in exploring a particular subject, an unwary beginner may imagine that pupils will almost automatically come to share his joys and enthusiasms. The disappointments of the first year for such a neophyte may literally drive him from the classroom. Unless he can direct some of his intellectual energies to the reformulation of materials from his teaching field, he is likely to

be punished rather than rewarded by the responses of his charges. Teacher-education programs, I am afraid, have been disastrously unsuccessful in convincing prospective teachers of the potential intellectual rewards in the manipulation of content for particular audiences.

Who is to help the new teacher to communicate the excitement and relevance of his teaching field to pupils whose attitudes and social experience may be ill-equipped to respond to either the discipline or the pleasure of systematic study? Obviously, responsibility cannot be delegated to the substantive collegiate departments, for they are neither staffed nor inclined to take on the task. A possible answer is Conant's clinical professor of education, but master teachers of the sort Conant envisages can only introduce the apprentice to the problems he will face as a regularly employed teacher in another school. Presumably, the responsibility now is upon the professor of education, but scholars interested in methods of teaching are normally more concerned with other aspects of practice than the organization of content for particular classes. This is as it should be, for it is impossible for a single professor to anticipate the range and variety of pupil populations with which beginning teachers will be confronted. But the basic fact remains, unfortunately, that there are few specialists in schools or colleges who know anything worth teaching about how to organize history, literature, or biology for particular groups of pupils. I believe that groups of practicing teachers, working in colleagial association in individual schools or at least in individual school systems, must inquire into the problem.

Let me be very clear on this point. I am definitely not suggesting that individual schools make such loose and fanciful translations as to lose sight of the original text. If we have learned anything from the new curriculum movement, it is that

> every child, no matter what his measured capacity to learn, has the right to direct scholarly experience in the sciences, the social sciences, and the humanities. But every child's experience of scholarly inquiry poses new questions and new problems for the teacher. No matter what the level of detail of curricular packages, their full value can never be reached without the intervention, the thoughtful translation if you will, of a wise teacher. Even if it were possible for external curriculum writers to do the full job, I would resent the teacher's exclusion from his best chance of intellectual delight. Intelligently conceived curriculum materials, of course, need not attempt to impose a rigid uniformity but can deliberately build in modes and possibilities for departure by the individual teacher. And, finally, as I have earlier asserted, how can children fully know the dynamism of learning if the adults around them stand still?[100]

Dean Schaefer's remarks are very much in the spirit of the Madison Project's approach to curriculum evolution, and shed some light on the difficulties of viewing so flexible an undertaking through the prism of the conventional wisdom of educational theory.

11. Sheldon Gold, "Graphing linear equations -- a discovery lesson," The Arithmetic Teacher, Vol. 13, No. 5 (May, 1966), pp. 406-407. Returning to the mysteries of the diffusion of innovation, there is no uncertainty in this case. Sheldon Gold has both studied with the Project, and worked with it in conducting "big city" workshops.

12. Russell C. Magnuson, "Signed numbers," The Arithmetic Teacher, Vol. 13, No. 7 (November, 1966), pp. 573-575. Mr. Magnuson has worked with the Project, and presently does so full-time, as the Project's resident coordinator for cooperation with the Los Angeles City Schools.

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[100] From: The School as a Center of Inquiry by Robert J. Schaefer. Reprinted by permission of the publisher, Harper and Row.

13. Louis S. Cohen, "A rationale in working with signed numbers -- revisited," The Arithmetic Teacher, Vol. 13, No. 7 (November, 1966), pp. 564-567. Again there is no mystery; Louis Cohen has worked for the Project.

14. Donald Cohen, "Inquiry in mathematics -- with children and teachers," The Arithmetic Teacher, Vol. 14, No. 1 (January, 1967), pp. 7-9. Here, too, the lines of diffusion are easy to trace; Donald Cohen also has worked for the Project, and is presently resident coordinator for the Project in New York City.

15. Raymond Sweet, "Organizing a mathematics laboratory," The Mathematics Teacher, Vol. LX, No. 2 (February, 1967), pp. 117-120.

Raymond Sweet has no known connection with the Project, and appears to be pursuing somewhat similar lines entirely independently.

16. Rosemary C. Anderson, "Let's consider the function!" The Arithmetic Teacher, Vol. 14, No. 4 (April, 1967), pp. 280-284. This article specifically refers to the Madison Project. (From a "diffusion of innovation" point of view the lines of communication are not known.)

17. S.E. Sigurdson and Halia Boychuk, "A fifth-grade student discovers zero," The Arithmetic Teacher, Vol. 14, No. 4 (April, 1967), pp. 278-279. This article also refers to use of Madison Project material in the classroom. The lines of communication are not known.

Anyone interested in the "diffusion of innovation" aspects of the Project's work may wish to study quarterly reports, and other records, available from The Madison Project, 8356 Big Bend Blvd., Webster Groves, Missouri 63119.

Finally, three reports of different kinds:

18. Barbara Maury Bowen, Patterns, Georgia State Department of Education, 1967.

19. Mary Lela Sherburne, The Cardoza Model School District: A Peach Tree Grows on T Street. (Education Development Center, Inc., 55 Chapel Street, Newton, Massachusetts 02158) (1967). This discusses classroom use of Madison Project materials in the Model School District of the Washington, D.C., Public Schools.

20. Professor Carol Kipp, of U.C.L.A., has some results from administering test items on mathematical content knowledge to teachers in various situations, including several hundred who had attended Madison Project workshops. For information, write to Professor Carol Kipp, Department of Education, 405 Hilgard Ave., University of California, Los Angeles, California 90024.

This list is deliberately both incomplete and somewhat random. A very large number of informal reports on classroom use of Madison Project materials are now in the literature, and there are doubtless many studies -- possibly some quite important ones -- which the Project has never heard about. It is hoped that the Project's materials and approach -- and possibly even its films -- will be used in more studies in the future.

We began this Appendix with some remarks on "evaluation" and description, the "conventional wisdom" of educational theory, and the issue of artist-practitioner (or even "primitive" native) vs. the "scientific" theorist, and we now close with this theme, with four remarks:

Remark 1: The conventional wisdom suggests that one must evaluate student performance. A novel approach to this, in the case of mathematics, has been developed by Professor Richard Singer and Professor Katharine Kharas of Webster College, and applied to Webster College mathematics majors. The idea is borrowed from schools of art: each student over his four years at the College accumulates a "folio" of his work in mathematics. The student shapes this as he chooses, although -- as in art -- there is some faculty influence. A student may make up his own axiom system and prove theorems within it. He may solve problems that he himself devises, or he may take a standard treatise and solve problems in it. He may work out a mathematical theory for a physical phenomenon and compare his theoretical results with the laboratory data obtained by actually performing relevant experiments. He may use statistics in studying the College itself, or some other phenomenon -- such as the ability of certain statistical data to identify correctly the book from which a sample page was selected. He may learn to program a computer, and then write an original program to play some game, etc. (Cf. the work with computer programming by 12th graders described by Wallace Feurzeig of Bolt, Beranek, and Newman.)

In any event, this cumulative folio of his own personal work is the student's record and "proof" of his achievement.

This device has great and subtle merits. In an age that threatens to take away from each individual a larger part of himself, to deny his individuality, to impose limitations which are becoming increasingly gratuitous, to invade his privacy, his freedom, his mind, his psyche, and his soul, to cut him off from effective control of his own society, we need urgently to find ways to give something of a person back to himself. He would surely appear to be the rightful owner.

The Singer-Kharas "folio" evaluation surely seems to give the student a larger role in shaping his own life than standardized tests will usually do.

Remark 2: The Nuffield Project has replaced the idea of "achievement tests" by the quite different idea of "check points," developed somewhat in the spirit of Jean Piaget.

However, even here there is the distinction, as David Hawkins has pointed out, between conceptualizing education as a journey along a path, or as a romp around a field. In the latter case, there is no reason to suppose that any particular check-point ever will be passed, and the loss may not be tragic. Many people have never learned to play the contra-bassoon, and are not necessarily much the worse for it.

Remark 3: How do formal evaluations compare with intuitive perceptions in the specific case of classroom teaching? We have spoken earlier of the large amount of feedback data available when curricula are developed in the classroom, with the students themselves as active participants. An example may be useful. A class of 5th grade children (at the beginning; they were 7th graders at the end) was used in the Chicago "big city" workshops as a demonstration class for teacher education; this meant that they came to school -- voluntarily -- on Saturdays and during the summer. The teacher of this class gradually realized that for two years these children had been going to school six days a week, and during part of the summer -- voluntarily. Morale, in fact, was beginning to wear thin, and it appeared that this arrangement might not last much longer.

The teacher had been standing at the chalkboard much of the time, and often dominating class discussions. He now broke the class up into groups of three or four children each, seated around tables, using physical materials much of the time, with each group working independently of the others. Morale improved dramatically. Several hundred teachers were viewing this demonstration class over closed-circuit TV; they had not been told about this decision, or the reason for it, yet many spontaneously reported that the improvement in morale was so dramatic that it could be felt over the TV. This coincided precisely with the judgement of the class teacher. (Incidentally, the students never did desire the end of the program, and it was terminated only when it became unfeasible to continue it.)[101]

This result also confirmed the independent interview study conducted by Herbert Barrett with students in Connecticut (and discussed earlier in this Appendix). But -- suppose it had not?

Suppose a relatively formal study contradicted some strongly-held teacher beliefs, that had been extensively discussed and were widely shared.

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[101] In this case the same students were involved for about two years; it should be born in mind that many students have been in Project classes for as long as five consecutive years, which means that the Project is able to observe rather long-term trends.

Which would you believe?

Remark 4: Finally, do we need to make judgements on individual students? On the one hand, selection between students becomes less important as college education becomes increasingly available, and as "weaker" colleges are becoming stronger and stronger.

Again, is competition between individuals really as necessary today as it once was -- or even as desirable?

Still further: the priest or lawyer cannot be called upon to testify against his parishioner or client. Should a teacher "testify" against a student? In every case the patron came seeking some professional assistance. The asymmetry of the relationship is highly suggestive: perhaps teachers should not report on what students know or have learned. If someone has legitimate reason to inquire into such matters, perhaps that should be their problem, and they should solve it without violating the privacy of teacher-student relations.

The conventional wisdom of educational theory has hardly examined such questions. This would seem to involve assuming rather a lot, and doing it quite uncritically.

Appendix B

Partial List of Madison Project Films.

A large number of prototype lessons taught in the implementation program described in this report were recorded on videotape, for subsequent transfer to 16 mm. sound motion picture film. The financial support for this recording procedure was provided by Webster College, and by the Course Content Improvement Section of the National Science Foundation. The development of the prototype lessons themselves was of course part of the present U.S.O.E. program.

The N.S.F. film project is presently scheduled for completion on approximately June 30, 1968. At the present stage it is possible to record a partial list of prototype lessons that should appear on these films.

Obviously, each lesson can be regarded from various points of view, and hence many different schemes for classifying these lessons could be devised. Perhaps the most useful, for present purposes, is a division into the following six categories, which are not mutually exclusive:

I. Lessons emphasizing small-group work and individualized instruction, grades K-8.

II. Lessons intended to improve the students' understanding of topics in traditional arithmetic, grades K-8.

III. Lessons concerned with creating a "bridge" or "foundation" for unifying arithmetic, algebra, and geometry in grades K-8.

IV. Lessons concerned with creating a "bridge" or "foundation" for relating mathematics to science in grades K-8.

V. Lessons intended to give more capable students a head start on high school and college mathematics.

VI. Lessons from the Ninth-Grade Course.

1. Lessons Emphasizing Small-Group Work and Individualized Instruction, Grades K-8.

Title	Video-tape no.	Grade level	Teacher	Content
Primary Grades (K-2)				
1. (not yet titled)	VT 34, part 2	Kindergarten	Doris Machtinger	Place-value numerals via physical materials
2. Teaching Big Ideas in Mathematics to First Grade Pupils -- Lesson 1	VT 71	Grade 1	Beryl S. Cochran	Counting, addition of whole numbers, introduction to the number line, plotting points in first quadrant in Cartesian coordinates, geoboards, place value numerals via physical materials.
3. Teaching Big Ideas in Mathematics to First Grade Pupils -- Lesson 2	VT 72	Grade 1	Beryl S. Cochran	Work with geoboards; place-value numerals via concrete materials; plotting points in the first quadrant in Cartesian coordinates.
4. Teaching Big Ideas in Mathematics to First Grade Pupils -- Lesson 3	VT 73	Grade 1	Katherine Vaughn	The concept of height and the process of measuring height.
5. Teaching Big Ideas in Mathematics to First Grade Pupils -- Lesson 4	VT 74	Grade 1	Katherine Vaughn	Continuation of lesson begun on VT 73.
6. Teaching Big Ideas in Mathematics to First Grade Pupils -- Lesson 5	VT 75	Grade 1	Beryl S. Cochran	Place-value numerals via concrete materials plotting points in the first quadrant in Cartesian coordinates.
7. Addition and Multiplication Using Plastic Washers	VT 67	1st and 2nd graders	R. B. Davis	Addition and multiplication using plastic washers.
8. Multiplication Using Dots	VT 67	1st and 2nd graders	R. B. Davis	Multiplication using arrays of dots marked on paper.
9. Geometry via Concrete Objects	VT 76, Part 1	Grade 2	Beryl S. Cochran	Geoboards, E.S.S.'s "geoblocks," and E.S.S.'s "colored polygonal blocks."

Title	Video-tape no.	Grade level	Teacher	Content
10. Gluing and Stamping	VT 76, Part 2	Grade 2	Beryl S. Cochran	Further work with blocks, leading to concepts of volume, area, and similarity.
11. Multiplication Arrays and Other Topics	VT 36, Part I	Grade 2	Beryl S. Cochran	Use of rectangular arrays in multiplication; work with Cuisenaire rods; work with number line, including fractions.
12. Using Geoboards with Second Graders	VT 77, Part II	Grade 2	Katherine Vaughn	Mainly the concept of area, using geoboards.
Intermediate Grades (4-6)				
13. An Introduction to Geometry via Nailboards	VT 68	Mixed: children from grades 4, 5, and 6	R. B. Davis	The concept of area, including informal heuristic demonstrations.
14. A Sixth-Grade Lesson on Place-Value Numerals	VT 73	6th graders	R. B. Davis	Place-value numerals using concrete materials (including non-decimal numerals).
15. Measurement via Inequalities	VT 79	5th and 6th graders (mixed)	R. B. Davis	Measurement of length using arbitrary units (with fractional sub-units also available), obtaining inequalities as results.
16. The Classroom Divided into Small Groups: Counting, Volume, and Rational Approximations	VT 82	5th and 6th graders (mixed)	R. B. Davis	Counting large numbers; the concept of volume; and finding rational approximations to $\sqrt{2}$.
17. The Concepts of Volume and Area	VT 35	5th and 6th graders (mixed)	R. B. Davis	This lesson shows children's cognitive and perception difficulties in coping with the concepts of volume and area when physical materials are involved.

Title	Video-tape no.	Grade level	Teacher	Content
18. The Classroom Divided into Small Groups: Volume and Area	VT 88	5th and 6th graders (mixed)	R. B. Davis	Work with physical materials related to the concepts of volume and area.
19. Small-group Instruction: Signed Numbers, Rational Approximations, and Motion Geometry	VT 84	6th graders	R. B. Davis	Work with the arithmetic of signed numbers, rational approximations to $\sqrt{2}$, and motion geometry leading to binary operation tables for groups.
20. Small-group Instruction: Committee Report on Signed Numbers	VT 87, Part I	6th graders	R. B. Davis	The arithmetic of signed numbers.
21. Small-group Instruction: Committee Report on Rational Approximations	VT 87, Part II	6th graders	R. B. Davis	Rational approximations to $\sqrt{2}$.
22. Small-group Instruction: Committee Report on Motion Geometry	VT 87 Part III	6th graders	R. B. Davis	Binary operation tables obtained from motions of symmetric plane figures.
23. A Wide-Angle View of a Math Lab: A New Role for Teachers?	VT 100	6th graders	Gerald B. Glynn	This film merely shows the over-all re-arrangement of a classroom to make it suitable for use as a "math lab."
24. Math Labs for the Intermediate Grades --A Discussion	VT 101	not a lesson		
Teacher Education				
25. Teachers Studying Geoboard Geometry	VT 81	The "students" are elementary school teachers	Irvin Vance	The use of geoboards in constructing informal heuristic demonstrations dealing with the area of plane figures.

II. Lessons Intended to Improve the Students' Understanding of Topics in Traditional Arithmetic, Grades K-8

Counting: Primary Grades (K-2)

Title	Video-tape no.	Grade level	Teacher	Content
Films 1, 2, and 3 from Section I, plus:				
35. The Operations of Arithmetic Using Concrete Materials	VT 38	Grade 2	Beryl S. Cochran	Experiences sharing, distributing, combining, and removing discrete physical objects.
36. Subtraction and Division	Excerpt from Film #35.			

The Concepts of Addition, Subtraction, Multiplication and Division

Primary Grades (K-2)

Films number 2, 3, 7, 8, and 11 from Section I; plus films #35 and 36 from Section II.

Place-Value Numerals, and Algorithms Related to the Use of Place-Value Numerals

Primary Grades (K-2)

Films number 1, 2, 3, and 6 from Section I.

Intermediate Grades (4-6)

Film number 14 from Section I; also

61. Kye's Arithmetic	VT 23, Part 2	Grade 4	R. B. Davis	Algorithms for arithmetic using negative digits.

Title	Video-tape no.	Grade level	Teacher	Content
Readiness for Fractions, and Understanding Fractions				
Primary Grades (K-3)				
Films number 9, 11, and 12 from Section I; plus				
28. Experience with Fractions--Lesson I	VT 14	Grade 2	R. B. Davis	Experience with the informal idea of fractional parts of discrete collections of physical objects (actually, "kindergarten blocks").
29. Experience with Fractions--Lesson II	VT 21	Grade 2	R. B. Davis	Sequel to film #28.
30. The Number Line and Other Topics	VT 22	Grade 2	Beryl S. Cochran	Using the number line to aid in solving equations, experience with fractions, the concept of identity (i.e., universally-true open sentences), Cartesian coordinates.
32. Experience with Fractions, Number Line, and String	Excerpt from film #30			
Intermediate Grades (4-6)				
Film number 15 from Section I; plus				
51. Dividing Fractions	VT 15, Part I	Grade 4	R. B. Davis	Division of Fractions
52. Excerpt from "Dividing Fractions"	Excerpt from film #51			

Using Arithmetic

Intermediate Grades (4-6)

Films number 15, 16, 17, 18, 19, 21, and 23 from Section I.

III. Lessons Concerned with Creating a "Bridge" or "Foundation" for Unifying Arithmetic, Algebra, and Geometry in Grades K-8.

See also films number 2, 3, 6, 9, 10, 12, 13, 15, 16, 17, 18, 19, 20, 21, 23, 24, and 25 from Section I.

Title	Video-tape no.	Grade level	Teacher	Content
Primary Grades (K-3)				
26. A Lesson with Second Graders	VT 4	Grade 2	R. B. Davis	Introduction to the concepts of signed numbers, the number line, open sentences, and Cartesian coordinates.
27. Pebbles in the Bag	Excerpt from film #26			Signed numbers.
28. Experience with Fractions -- Lesson I	VT 14	Grade 2	R. B. Davis	Experience with the informal idea of fractional parts of discrete collections of physical objects (actually "kindergarten blocks").
29. Experience with Fractions-- Lesson II	VT 21	Grade 2	R. B. Davis	Sequel to film #28.
30. The Number Line and Other Topics	VT 22	Grade 2	Beryl S. Cochran	Using the number line to aid in solving equations, experience with fractions, the concept of identity (i.e., universally-true open sentences), Cartesian coordinates

Title	Video-tape no.	Grade level	Teacher	Content
31. Open Sentences and the Number Line	Excerpt from film #30			
32. Experience with Fractions, Number Line, and String	Excerpt from film #30			
33. Experience with Identities	Excerpt from film #30			
34. Tic Tac Toe in Four Quadrants	Excerpt from film #30			
35. The Operations of Arithmetic Using Concrete Materials	VT38	Grade 2	Beryl S. Cochran	Experiences sharing, distributing, combining, and removing discrete physical objects.
36. Subtraction and Division	Excerpt from film #35			
37. Graphs and Truth Sets	VT 42	Grade 2	Katherine Vaughn	Open sentences and Cartesian coordinates.
38. Crossed Number Lines	Excerpt from film #37			
39. The Graph of $\square + \triangle = 8$	Excerpt from film #37			
40. The "Rule for Substitution"	Excerpt from film #37			
41. Operations of Arithmetic	VT 58	Grade 1	Beryl S. Cochran	Experiences with arithmetic using concrete materials.
42. Excerpt from "Operations of Arithmetic"	Excerpt from film #41			

Title	Video-tape no.	Grade level	Teacher	Content
Intermediate Grades (4-6)				
43. First Lesson	VT A-1	Ungraded class with children in grades 3 through 7	R.B. Davis	Introduction of the concepts of variables, open sentences, signed numbers, and Cartesian coordinates.
44. Discovery	Made from part of VT A-1	Includes narration		Discussion of part of VT A-1
45. Sequencing and Elementary Ideas	Made from part part of VT A-1	Includes narration		Discussion of part of VT A-1
46. Excerpt on True, False, and Open Sentences	Excerpt from film #43			
47. Second Lesson	VT A-2	Ungraded class with children in grades 3 through 7	R.B. Davis	Linear graphs and identities (i.e., universally-true open sentences)
48. Introduction to Identities	Excerpt from film #47			
49. Balance Pictures	VT 8	Grade 4	R. B. Davis	Heuristic approach to solving linear equations.
50. Guess My Rule	VT 12	Grade 4	R. B. Davis	Given a table of ordered pairs, find a corresponding algebraic formula.
51. Dividing fractions	VT 15 Part 1	Grade 4	R. B. Davis	Division of fractions.

Title	Video-tape no.	Grade level	Teacher	Content
52. Excerpt from "Dividing Fractions"	Excerpt from film #51			
53. Experience with Area	VT 15 Part 2	Grade 4	R. B. Davis	Introduction of the concept of area.
54. Experience with Estimating and Measuring Angles	VT 16	Grade 4	Betty Bjork	Experience with estimating and measuring angles.
55. Experience with Angles	VT 29 Part 2	Grade 5	R.B. Davis	The concept of angle as a measure of rotation; use of physical materials such as wheels, etc.
56. Units of Measurements: Distance and Angles	VT 49 Part 2	Grades 6 and 7 (mix (mixed)	R. B. Davis	Selection of appropriate physical objects to serve as concret "units" for measuring distance and angles.
57. Some Aspects of Measurement	Excerpt from film #56			
58. Weights and Springs	VT 18 Part 2	Grade 6q	R.B. Davis	A study of elasticity and the recognition of linear functions.
59. Analysis of Classroom Behavior	Made from film #58, in order to emphasize the more effective use of films showing classroom lessons.			
60. Experience with Linear Graphs	VT 23 Part 1	Grade 4	R.B. Davis	Graphs of linear functions.

Title	Video-tape no.	Grade level	Teacher	Content
61. Kye's Arithmetic	VT 23 Part 2	Grade 4	R.B. Davis	Algorithms for arithmetic using negative digits.
62. Postman Stories	VT 45	6th and 7th graders (mixed)	R.B. Davis	The arithmetic of signed numbers, and the graph of a circle.
63. Circle and Parabola	VT 24	Grade 6	R. B. Davis	Graphing conic sections by plotting points.
64. Graphing a Parabola	Edited excerpt from film #63			
65. A More Formal Approach to Variables	VT 28 Part 1	Grade 4	R.B. Davis	Consideration of the replacement set for a variable
66. Guessing Functions	VT 49 Part 1	6th and 7th graders (mixed)	R.B. Davis	Given a table of ordered pairs, find a corresponding algebraic formula.
67. A Week of Mathematical Exploration -- Monday	VT 60	4th and 5th graders (mixed)	R.B. Davis	Initial introduction to the arithmetic of signed numbers, variables, open sentences, and plotting points in Cartesian coordinates.
68. A Week of Mathematical Exploration -- Tuesday	VT 61	4th and 5th graders (mixed)	R.B. Davis	Arithmetic of signed numbers, introduction to the concept of functions.
69. Excerpt on "Making up a Rule"	Excerpt from film #68			

Title	Video-tape no.	Grade level	Teacher	Content
70. A Week of Mathematical Exploration -- Wednesday	VT 62	4th and 5th graders (mixed)	R.B. Davis	Functions
71. A Week of Mathematical Exploration -- Thursday	VT 63	4th and 5th graders (mixed)	R.B. Davis	Continuation of topics above.
72. A Week of Mathematical Exploration -- Friday	VT 64	4th and 5th graders (mixed)	R.B. Davis	Continuation of topics above.
73. Introduction to Postman Stories	VT 69 Part 2	4th, 5th, and 6th graders (mixed)	Katharine Kharas	Introduction to the arithmetic of signed numbers.
74. Linear Measurement	VT 80	Grade 6	R.B. Davis	Linear measurement using inequalities.
75. Average and Variance	VT 17	Grade 6	R.B. Davis	Statistical attributes of measurement variability.

IV. Lessons Concerned with Creating a "Bridge" or "Foundation" for Relating Mathematics to Science in Grades K-8.

Essentially all of the films of Section III, plus:

Title	Video-tape no.	Grade level	Teacher	Content
95. Velocity and Acceleration	VT 32 and VT 33	Grade 8	R.B. Davis	Slope of a curve, graphical differentiation, graphical integration, and the study of displacement, velocity, and acceleration, using PSSC experiments.
96. Jeff's Experiment	A short excerpt from film #95.			
98. Yarn-Breaking Experiments	VT 1 and VT 2	Grade 7	R.B. Davis	Using vector addition to study the breaking of yarn under various conditions of loading.
99. Forces and Vectors	Excerpt from film #98.			

V. Lessons Intended to Give More Capable Students a Head Start on High School and College Mathematics

[See also films number 19 and 22 in Section I.]

Primary (K-3)

The performance of the 3rd grade girl named Ruth, in the film Second Lesson, gives one suggestion as to what "more advanced concepts" might mean for the 3rd grade curriculum.

Title	Video-tape no.	Grade level	Teacher	Content
76. Clues	VT 13 Part 1	Grade 6	R.B. Davis	Implication, contradiction, and uniqueness.
77. Clues -- Audience Participation Version	An edited version of film #76			

[This subject is also dealt with at the 9th grade level; cf. film #104.]

Matrices

78. Matrices	VT 7 Part II	Mixed 5th and 6th graders	R.B. Davis	Exploration of the structure of the system of 2-by-2 matrices
79. Solving Equations with Matrices	VT 13 Part II	6th graders	R.B. Davis	Using an isomorphism between rational numbers and a subset of the set of 2-by-2 matrices in order to solve the equation $x^2 = {}^{-}4$.

[For a continuation of the subject of matrices into junior high school, see film #92.]

Identities and Derivations

80. Accumulating a list of Identities	VT 18 Part I	Grade 6	R.B. Davis	Identities (i.e., universally-true open sentences)
81. Debbie's List	VT 28 Part 2	Grade 4	R.B. Davis	The distributive law expressed as an identity.

Title	Video-tape no.	Grade level	Teacher	Content
82. "Super-identities"	VT 59	Grade 5	Katharine Kharas	Introduction of additional variables into identities.
83. Introduction to Derivations	VT 9 1st part of Part I	Mixed 5th and 6th graders	R.B. Davis	The concept of implication applied to identities.
84. Axioms and Theorems	VT 10	Grade 6	R.B. Davis	Establishing some identities as theorems proved by using other identities.
85. Six Excerpts on identities	This film uses excerpts from films #80, 82, 83, 84, and 106.			
[This topic is also dealt with at the 9th grad level in films #104, 105, 106, 107, 115 and 116.]				
Area				
86. The Area of a Parallelogram	VT 9 Part 1	Grades 5 and 6 mixed	R.B. Davis	Informal heuristic demonstration of the formula for the area of a parallelogram.
Classification of Functions, Finite Difference Methods, etc.				
87. The Study of Functions -- Linear, Quadratic, and Exponential	VT 66	Grades 4, 5, and 6 (mixed)	R.B. Davis	Finite difference methods for distinguishing between linear, quadratic, and exponential functions.

Title	Video-tape no.	Grade level	Teacher	Content
Motion Geometry and Binary Operation Tables for Groups				
See films #19 and #22.				
Programming Computers				
88. Programming the IBM 1620, Using GOTRAN	VT 3, Part 1	Grades 5 and 6 (mixed)	R.B. Davis	Programming the IBM 1620, using GOTRAN.
Word Problems				
89. Word Problems	VT 9, Part 2	Grades 5 and 6 (mixed)	R.B. Davis	Word problems and the square root of 2.
Derivation of the Quadratic Formula				
90. Derivation of the Quadratic Formula -- 1st Beginnings	VT 3, Part 2	Grades 5 and 6 (mixed)	R.B. Davis	Derivation of the quadratic formula.

[For a continuation of this topic into Junior High School, see film #91.]

[This topic is dealt with also at the 9th grade level; see film #110.]

Title	Video-tape no.	Grade level	Teacher	Content
Grades 7 and 8				
Derivation of the Quadratic Formula				
91. Derivation of the Quadratic Formula -- Final Summary	VT 6, Part II	Grade 7	R.B. Davis	Derivation of the quadratic formula.
[This topic is dealt with also at the 9th grade level; see film #110.]				
Matrices				
92. Complex Numbers Via Matrices	VT 6, Part 1	Grade 7	R. B. Davis	Exploration of the structure of the system of complex numbers, using matrix representation.
Logic				
93. Introduction to Truth Tables and Inference Schemes	VT 19, Part 1	Grade 7	R.B. Davis	Introduction to truth tables and inference schemes.
94. Logic	VT 56 and VT 57	Grade 8	R. B. Davis	Truth tables.
[This topic is dealt with also at the 9th grade level; see films #104, 105, 106, 115 and 116.]				

Title	Video-tape no.	Grade level	Teacher	Content
Velocity and Acceleration				
95. Velocity and Acceleration	VT 32 and VT 33	Grade 8	R. B. Davis	Slope of a curve, graphical differentiation, graphical integration, and the study of displacement, velocity, and acceleration, using PSSC experiments.
96. Jeff's Experiment	A short excerpt from film #95.			
Graphs of Functions in Cartesian Coordinates				
97. Graphing an Ellipse	VT 7, Part 1	Grade 7	R.B. Davis	Graphing $x^2 + k^2y = 25$, and studying the effect of the parameter k.
Statics				
98. Yarn-Breaking Experiments	VT #1 and VT #2	Grade 7	R.B. Davis	Using vector addition to study the breaking of yarn under various conditions of loading.
99. Forces and Vectors	Excerpt from film #75			
Probability				
100. Experience with Empirical Probability	VT 19 Part 2	Grade 7	R.B. Davis	Experience with empirical probability.

Title	Video-tape no.	Grade level	Teacher	Content
Limit of a Sequence				
101. Limits (1st Version)	VT 25	Grade 8	R.B. Davis	Three lessons on limits, using the ϵ, N definition
102. Limits (2nd Version)	VT 31	Grade 8	R. B. Davis	
103. Limits of Simple Sequences	VT 26	Grade 8	Professor Ross Finney	
VI. Lessons from the Ninth-Grade Course				
104. Identities, Proof, and Implication	VT 35	Grade 9	R.B. Davis	Proofs of some identities from other identities, and David Page's game of "Hidden Numbers" used to illustrate the concepts of implication, contradiction, and uniqueness.
105. Theorems on Additive Inverses, and a Study of Truth Tables	VT 36	Grade 9	R.B. Davis	Proof of the theorem $^{\circ}(^{\circ}A) = A$; subsequent consideration of the logical structure of the proof; and a representation of truth tables as mappings of one space into another.
106. Negative 1 Times Negative 1 Equals Positive 1	VT 37	Grade 9	R.B. Davis	Consideration of the theorem $^{-}1 \times {}^{-}1 = {}^{+}1$, from three different points of view.

Title	Video-tape no.	Grade level	Teacher	Content
107. Three Approaches to Signed Numbers. 1. The Axiomatic Approach	Excerpt from film #106			
108. Three Approaches to Signed Numbers. 2. Postman Stories	Excerpt from film #106			
109. Three Approaches to Signed Numbers. 3. Functionals	Excerpt from film #106			
110. Quadratic Equations	VT 43	Grade 9	R.B. Davis	Derivation of the quadratic formula
111. Introduction to Infinite Sequences	VT 44	Grade 9	R.B. Davis	Monotonic sequences related to $\sqrt{2}$.
112. What is Convergence?	VT 46	Grade 9	R.B. Davis	This continues the study of infinite sequences.
113. Bounded Monotonic Sequences	VT 47	Grade 9	R.B. Davis	The study of bounded monotonic sequences.
114. Introduction to the Complex Plane	VT 48	Grade 9	R.B. Davis	Matrix names are assigned to points in E_2.
115. A Language for Algebra	VT 53	Grade 9	R.B. Davis	Logic used as a language for the discussion of axiomatic algebra.

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Title	Video-tape no.	Grade level	Teacher	Content
116. Algebraic Systems: Intuition and Formal Descriptions	VT 55	Grade 9	R.B. Davis	Consideration of the real number system, of the system of 2 x 2 matrices, and of Boolean algebra.
117. A Lesson in Euclidean Geometry	VT 39 and VT 40	Grade 9	Elizabeth Herbert	Proofs of theorems in Euclidean geometry.

Appendix C

A Partial List of Actual Classroom Lessons Recorded on Audio-tape.

During the academic year 1959-1960, the Madison Project began audio-taping actual classroom lessons. Following the tapes and films over the years shows a clear evolution of Project ideas concerning curriculum and instruction. However, by far the largest number of tapes -- and those with the best technical quality -- were made in Weston, Connecticut between autumn, 1959 and spring, 1964, under the technical leadership of Morton Schindel, President of Weston Woods Studios, through the cooperation of school principal Gilbert Brown, and with Mrs. Beryl S. Cochran serving as sound engineer. Most of these tapes are stereophonic. Professor Richard Alpert of Harvard University, and the Brown Sound Company of Syracuse, New York, served as consultants and advisors. The financial support for making the audio-tape recordings of classroom lessons was provided by the Marcel Holzer Foundation, and by the Alfred P. Sloan Foundation. The curriculum involved is the highly sophisticated and unhappily unstable curriculum designated as "Curriculum ϵ ."

In some cases it is possible to follow the same children for as long as five years. Many of the same classes appear on audio-tape and also on film. Where this is the case, the classes are identified by the letters assigned to them in the 1965 U.S.O.E. Final Report, namely:

Robert B. Davis, A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere. (The Madison Project), 1965, pages 55-84. (This report will be referred to below as "1965 USOE Report.")

Chicago classes at Admiral Richard E. Byrd School were recorded by the courtesy of Kenneth Kobukata.

With one or two exceptions, the following list of tape-recorded lessons is arranged by classes of children, so that it is possible to follow the same children over, in many cases, a period of several years.

Sequence I

The class involved here is identified in the 1965 USOE report as "Class A."[102] In films, the nametags of these children read "Lex," "Bruce," "Ann," "Sarah," "Debby H.," "Ellen," "Geoff," "Jeff," etc. Madison Project lessons with Class A began in the academic year 1959-1960, when the children were 5th graders, and ended with academic year 1963-1964, when they finished grade 9. The class was in Weston, Connecticut.

The following listing of some tape recorded lessons has been selected essentially randomly; all Project lessons with this class, except for a few of the earliest ones, were recorded either on audio-tape or else on video-tape, and all tapes have been preserved.

1. Audio-tape No. 13. Lesson taught March 28, 1960. (The children at this point are, of course, 5th graders.) This is an early lesson on derivations. As was the Project's practice at the time, the teacher used certain of the students' abilities in "implicit" or "unexamined" form, and only made things explicit when the time to do so seemed to have come. Prior to this lesson there had been no interpretation of addition or multiplication as "binary" operations, and no explicit attention paid to either the associative law for addition ("ALA") or to the associative law for multiplication ("ALM"). All of this had been left implicit. In this particular lesson, ALA and ALM were introduced to these children explicitly for the first time. (Cf. Robert B. Davis, Discovery in Mathematics: A Text for Teachers.[103] Addison-Wesley, 1964, p. 188, and especially p. 215. A similar tacit use of certain intuitive ideas,

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[102] It should be mentioned that Classes A, B, and C were in Weston, Connecticut, at a time when Weston used very carefully (and wisely) arranged homogeneous grouping, on a "three track" plan. In autumn, 1959, Class A was the top track (most gifted) 5th grade class (out of three), Class B was the top track 4th grade class, and Class C was the top track 3rd grade class. Weston itself was, of course, an above-average community. The mean I.Q.'s for the top track classes were not especially high, howhowever, running around 122 to 124.

[103] Throughout the present Appendix, this volume will be referred to as "Discovery."

in the case of the distinction between rational and irrational numbers, is mentioned on p. 111 of this same volume. Max Beberman and others have pointed out that it is possible to interpret addition as something other than a "binary operation.")

2. (unnumbered) Lesson taught April 11, 1960.

The subject of the lesson is identities (cf. Discovery, Chapters 5, 6, 28, 29, 30, and 32).

3. Audio-tape No. 44. Lesson taught April 18, 1960.

Proof of the theorem: $(3 \times \square) + (2 \times \square) = 5 \times \square$. (Cf. Discovery, Chapter 33.)

4. Audio-tape No. 53. Lesson taught May 9, 1960.

Subject of lesson: word problems.

5. Audio-tape No. 63. Lesson taught May 16, 1960.

Content of lesson:

i) Derivation of the theorem: $3 - 2 = 1$

ii) Derivation of the theorem: $^{\circ}(A \times B) = (^{\circ}A) \times B$

iii) Some word problems.

6. Audio-tape No. 122. Lesson taught September 22, 1960.

The children are, of course, now beginning grade 6.

Contents of lesson:

i) Finding the identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

(Cf. Robert B. Davis, Explorations in Mathematics: A Text for Teachers, 1967, Chapter 37.)[104]

ii) Linear transformations on vectors ("vectors" are defined merely as "column matrices").

iii) basis vectors

iv) consideration of the task of finding matrix inverses.

7. Audio-tape No. 132. October 13, 1960.

Contents of lesson:

i) (review) multiplication of matrices

ii) observation (from examples) of the two facts that, if

$$\begin{pmatrix} M \end{pmatrix}$$

is a 2-by-2 matrix, then in

$$\begin{pmatrix} M \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

a knowledge of a and b determines the first column of M; and in

$$\begin{pmatrix} M \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}$$

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[104] This book will henceforth be referred to merely as "Explorations."

a knowledge of c and d determines the second column of M.

iii) If T is a linear transformation and α is a scalar, then

$$T \alpha \vec{u} = \alpha T \vec{u} \quad ,$$

for all scalars $\underline{\alpha}$ and all vectors $\underline{\vec{u}}$.

iv) Similarly, $T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$.

(This is an attempt to attack the problem of finding matrix inverses. Cf., however, Appendix F.)

8. Audio-tape No. 162. November 10, 1960.

(The children are in 6th grade.)

First lesson on binomial products and factoring.

Introduction of exponents.

9. Audio-tape No. 202. January 12, 1961.

Finding a pictorial representation for vectors, which have previously appeared only algebraically, as column matrices in the study of matrix algebra.

10. Audio-tape No. 312. May 11, 1961.

This class, which has been for some time -- at least a year and a half -- positively brilliant, eager, and creative in their approach to abstract mathematics, is now beginning to show the same "seventh grade slump" that seems to appear in every class we have observed, whatever kind of program they were pursuing.

Contents of this lesson:

i) Proof of the theorem: $x^2 \cdot (x - y) = x^3 - x^2 y$

ii) Consideration of equations which are, or are not, solvable within a given number system. (The intent is to lead into a study of complex numbers, which we plan to introduce by means of matrices.)

11. Audio-tape No. 322. May 25, 1961.

Content: some work with poles, their shadows, and similar triangles. (Note: this lesson, taught in 1961, was an "abstract" lesson using only discussion and the chalkboard. It was not until the "big city" workshop in New York City in the summer of 1967 that the Madison Project incorporated Lauren Woodby's elegant "outdoor mathematics," which treats this empirically, using actual poles and their shadows. This is highly suggestive of the very sizeable evolutionary change in the Project's program over this length of time. The Woodby outdoor approach was used successfully with culturally-deprived urban 5th graders, and thereabouts, in downtown Brooklyn. This marks some of the difference between "Curriculum ϵ," of 1960-1961 vintage, and "Curriculum β" of 1967.)

Sequence II

The class involved in "Sequence II" is the class that is identified in the 1965 USOE report (p. 61) as "Class B." In films, the nametags of the children read: "Beth," "Jean-Anne," "Toby," "Mark," "Flint," etc. Madison Project lessons with Class B began in the academic year 1959-1960, when the children were 4th graders, and ended with academic year 1963-1964, when they finished grade 8. The class was in Weston, Connecticut. For films involving Class B, see the 1965 USOE report, pp. 61ff.

12. Audio-tape No. 62. May 16, 1960.

The children were in grade 4 at this time.

This lesson deals briefly with four different topics:

i) quadratic equations with signed number roots, for practice in the arithmetic of signed numbers (cf. *Discovery*, Chapter 21);

ii) the derivation of an algebraic theorem

iii) the topic referred to in the lesson as "machines": actually, formulas that yield solutions for the general form of some equation (cf. *Discovery*, Chapter 37);

iv) systems of simultaneous linear equations (cf. *Discovery*, Chapter 45).

13. Audio-tape No. 92. (undated, but during 1959-1960 academic year.)

Contents of lesson: *measurement*.

14. Audio-tape No. 151. October 27, 1960.

(The children at this point were in grade 5.)

Proof of the theorem: $A \times (B + C) = (C \times A) + (B \times A)$. This proof was inserted in the hope that it would aid the children in the rest of the lesson, which dealt with "machines," for such equations as:

$$(\square \times a) + (\square \times b) = c$$

and

$$(\square \times a) + \square = m .$$

(Cf. *Discovery*, Chapter 41.)

15. Audio-tape No. 161. November 10, 1960.

(The children at this point are in grade 5.)

Content:

i) axiomatic treatment of subtraction, including proof of the theorem: 3 − 2 = 1;

ii) "machines" (i.e., general formulas) for area (cf. *Discovery*, Chapter 50);

iii) lesson ends with teams playing the "matrix" game (cf. *Discovery*, Chapter 4).

16. Audio-tape No. 201. January 12, 1961.

(The children were in grade 5 at this time.)

Contents:

i) "machines" for area of a trapezoid

ii) proof of the theorem:

$$(\square + 2) \times (\square + 3) = (\square \times \square) + (5 \times \square) + 6.$$

Sequence III

The class involved in "Sequence III" is the class identified in the 1965 USOE report (p. 66) as "Class C." In films, the nametags of the children read: "Jeff," "Ricky," "Mary," "Pam," "Lilli," "Windy," "Jane," "Geoff," "Greg," "Kris," "Val," "Miklos," "Jill," "Judy," "Jackie," "Tina," "Don," "Jennifer," etc.

Madison Project lessons with Class C began in academic year 1959-1960, when the children were 3rd graders, and continued through academic year 1963-1964, when they completed grade 7. The class was in Weston, Connecticut. For films involving Class C, see the 1965 USOE report, pp. 66-68.

17. Audio-tape No. 164. November 10, 1960.

The children were in grade 4 at this time.

Contents of lesson: introduction of the idea of "machine" -- i.e., of general formula to solve all equations of a given type.

18. Audio-tape No. 194. January 5, 1961.

The children were then in grade 4.

Content: The class, divided into two teams, play the "identity game" -- i.e., Team A writes an open sentence on the board. Team B must either guess whether or not the sentence is an identity, or else they must challenge Team A's own ability to do so. If Team B elects to guess, and do so correctly, they earn ten points; if incorrectly, they earn nothing and lose nothing. If, however, Team B elects to challenge Team A, then if Team A can state correctly whether the sentence is or is not an identity, Team A earns twenty points; if Team A decides this incorrectly, then by so doing they cause Team B to earn twenty points.

Then it becomes Team B's turn to put an open sentence on the board, and the roles are reversed.

Whoever is ahead at the end of the allowed time, or else whoever earns two hundred points first, wins.

(This lesson format, which is worthwhile, does not appear in any of the Project's written materials prior to the present report. It does, however, appear in many tape-recorded lessons. It was used extensively in teacher education programs around 1961, but has gradually disappeared from use, probably mainly because so few people have time to study the tape-recorded lessons.

19. Audio-tape No. 204. January 12, 1961.

At this time the children were in grade 4.

Contents of lesson: "Balance pictures" (cf. Discovery, Chapter 24). The purpose of "balance pictures" is to provide some intuitive or preverbal experience that will, hopefully, build readiness for the idea of equivalent equations and transform operations. (Discovery, Chapters 23, 25, and 26.) The theory is that "balance pictures" provide some useful mental imagery in the sense of Tolman (Discovery, p. 144).

20. Audio-tape No. 311. May 11, 1961.

At this time the children were in grade 4.

Content: review of the arithmetic of signed numbers.

21. Audio-tape No. 321. May 25, 1961.

Contents:

i) quadratic equations with signed numbers

ii) systems of simultaneous equations such as

$$\begin{cases} \square + \triangle = 4 \\ \square - \triangle = 3. \end{cases}$$

(Cf. *Discovery,* Chapters 45 and 46.)

22. (unnumbered). Lesson taught March 5, 1962.

Content:

i) The game of making up and recognizing *identities,* described above (18. Audio-tape No. 194. January 5, 1961);

ii) The "word game," often seen in newspapers, where one starts with some word -- say, for example, "HELP" -- and one is to arrive finally at some other word (with the same number of letters) -- say "SING" -- under the two restrictions that you may change only one letter at a time, and that at every step you must have a correct English word. (The reason for considering such problems at this point is to prepare the students for *derivations,* which have a great deal in common with this word game.)

Remark. Merely reading over the preceding lesson descriptions -- if one is familiar with the actual tape-recorded lessons themselves -- makes certain aspects of "Curriculum $\in$ " stand out rather clearly. To start with, it is obviously very sophisticated, very abstract, and very verbal. Further, it was designed to build, and thereafter use, a certain kind of mental imagery. Whether it really does either build or use such imagery is, of course, another question, but if it does, this may account for important individual differences. Casual observation of people suggests that different individuals use imagery in quite different ways.

To shift, however, to what the teacher is doing, these lessons show rather clearly that he very often creates competitive situations between teams, with each individual student left free to retreat into the group, or to swing for a grandstand home run. Moreover, the games and competitions are so arranged that the central mathematical idea is central also to the playing of the game. This last condition is often lost sight of in contemporary "academic games." The game may (or may not) be exciting, but all too frequently the basic subject-matter ideas are merely peripheral to the game. These lessons are also conspicuous in the high degree of student autonomy, and in the respect which the teacher shows for each student. To borrow a phrase from Haim Ginott, "He treats them as if they are nobles."

23. (unnumbered) Lesson taught October 10, 1962.

The children at this time were in grade 6.

Contents:

i) The children make up a long list of identities.

ii) They then seek to sort them out into a set of Axioms, and a set of Theorems that can be proved from the axioms.

24. (unnumbered) Lesson taught October 24, 1962.

The children are in grade 6.

The lesson deals with axioms and theorems.

Sequence IV

As mentioned above, Classes A, B, and C were the top track classes in a three-track system. The rather remarkable success experienced in teaching them some extremely sophisticated mathematics raised the question of what might be accomplished with a middle or bottom track group. Consequently, a program of Madison Project lessons was arranged for some middle and bottom track groups. One class began with their first Madison Project lesson in May, 1960, at which time they were fourth graders, and represented a mixture of children drawn from the middle track and bottom track fourth grade classes. They continued during the following school year, but at one point -- as the records reproduced below will show -- Mrs. Beryl Cochran and Herbert Barrett regrouped them, eliminating a few children who did not seem to be benefiting much from the program, and who were becoming unruly.

25. Audio-tape No. 71. Lesson taught May 23, 1960.

This is the second Madison Project lesson for this class; at this time they are in grade 4.

Contents:

i) linear and quadratic equations

ii) identities

iii) the "Point Set" game -- actually, a modified version of the Japanese game of GO. (Cf. Discovery, Chapter 7.)

26. Audio-tape No. 81. June 6, 1960.

This is the third Madison Project lesson for these children, who are in grade 4.

Contents: identities.

27. Audio-tape No. 103. September 15, 1960.

The children are now fifth graders.

These are not precisely the same children who were in this class .

last spring. Hence this lesson was, in terms of content, the "classical first lesson" which can be seen on the film entitled First Lesson.

28. Audio-tape No. 163. November 10, 1960.

The children at this point were in grade 5.

Content:

i) "Postman Stories." (These are a "paradigmatic" or "mental imagery" approach to the arithmetic of signed numbers. Cf. Discovery, Chapters 12 and 14, or else Explorations, Chapters 5 and 6.)

ii) Quadratic equations

iii) Linear equations.

29. Audio-tape No. 193. January 5, 1961.

The children are in grade 5 at this time.

Between the lessons on tapes No. 163 and No. 193, Beryl Cochran and Herbert Barrett re-grouped the children, to eliminate some who did not seem to be profiting from the program, and who showed signs of becoming unruly.

Content of this lesson: "machines" (i.e., formulas) for such equations as:

$$a + \square = b$$

$$\square + a + b = w$$

$$(a \times \square) + b = c$$

(Discovery, Chapter 37.)

30. Audio-tape No. 203. January 12, 1961.

Contents:

i) The "identity game," discussed above (Sequence III, 18. Audio-tape No. 194).

ii) Discussion of $\frac{\square}{0} = \square$.

Sequence V

Considerations similar to those mentioned in relation to Sequence IV caused Mrs. Cochran to arrange for a "low-average" 3rd grade class to begin a sequence of Madison Project classes during the winter of 1960-1961.

31. Audio-tape No. 190-SPEC. January 5, 1961.

The children are in grade 3.

Content: this is a modified version of the Project's "classical first lesson" which can be seen on the film entitled First Lesson.

32. Audio-tape No. 200-SPEC. January 12, 1961.

Content: the "matrix game" (cf. Discovery, Chapter 4).

33. Audio-tape No. 210-SPEC. January 19, 1961.

Content: Linear equations such as:

$10 + \square = 17 \qquad \{{}^{+}7\}$

$10 + \square = {}^{-}5 \qquad \{{}^{-}15\}$

Sequence VI

This is a 6th grade class, in the "Middle School" in Scarsdale, New York, during the 1961-1962 academic year. These children had actually begun studying Madison Project material the year before (academic year 1960-1961).

One girl in this class is a 5th grader, transported from an elementary school to the "Middle School" specifically in order to to take part in the Madison Project classes; for the rest of the day and week she attends her regular elementary school. This girl had been identified because of her exceptional fine performance with Madison Project materials the previous year, when she had been a 4th grader.

34. (unnumbered) Lesson taught May 1, 1962.

Content: First lesson on graphing circles:

$$(\square \times \square) + (\triangle \times \triangle) = 25$$

$$(\square \times \square) + (\triangle \times \triangle) = 169$$

(For a comparison of the performance of students in Scarsdale, New York, with Negro children in the Banneker District in St. Louis, Missouri, cf. this audio-tape with the film entitled Postman Stories.)

Sequence VII

This, actually, is not a sequence, but rather a collection. In "Sequence VII" we do not follow the same children, over an extended period of time, as we do in all of the other sequences -- which is what justifies their name. In this collection of tape-recorded lessons, we jump around from one class to another, even from one school to another, in the neighborhood of Kampala, Uganda. These tapes were recorded as part of an E.S.I. project, financed through E.S.I., to test the appropriateness of materials from the United States for use in English-speaking equatorial Africa. This E.S.I. project expanded and became the Entebbe Mathematics Project (known also by various other names), and, having been very fully developed for use in Africa, it is now being imported for use in the United States. Over the years, six Madison

Project teachers have gone from the United States to work in school mathematics in Africa.

35. Tape designation: Africa No. 1.

Secondary I -- i.e., 9th grade, Makerere College School, Kampala, Uganda. Lesson taught September 6, 1961.

Content: "classical first lesson," parallel to the content of the lesson shown on the film entitled First Lesson.

36. Tape designation: Africa No. 5, Part II, continued onto Africa No. 6, Part I.

Secondary II -- i.e., 10th grade, Makerere College, School, Kampala, Uganda. Lesson taught September 6, 1961.

Content:

i) "List shortening" (Discovery, Chapters 27-30, or else Explorations, Chapter 20).

ii) The distributive law

iii) Proof of the theorem: $A \times (B + C) = (B \times A) + (C \times A)$.

37. Tape designation: Africa No. 6, Part II.

This class is also a "Secondary I" class (i.e., grade nine), but is not the same class as the one listed earlier in 35. This lesson, also, was at Makerere College School, and it was also taught on September 6, 1961. Here, however, the similarities end. Whereas virtually all African classes studied were very formal, very proper, very polite, and not very creative or venturesome mathematically (as judged by a teacher from the United States), this class responded much more like American children in Connecticut or Missouri. Perhaps the secret lies in the fact that the regular teacher of this class was, in fact, an American: a tall, relaxed young man from Minnesota named Larry Olds.

Content: "classical first lesson."

38. Tape designation: Africa No. 7.

Junior II A, Savio Junior Secondary School, Kisubi, Uganda. Lesson taught September 7, 1961.

39. (unnumbered). Junior IB Maths, Makerere College School, Kampala, Uganda. Lesson taught April 2, 1962.

Sequence VIII

As mentioned elsewhere in this report, in the summer of 1964 a class of supposedly culturally deprived children in Chicago, who were in fact very bright, were assembled from grades 4, 5, and 6 and combined into a single class to serve as a demonstration class, via closed-circuit TV, for the "big city" workshops in Chicago. These children voluntarily attended school for part of the summer of 1964, on Saturdays throughout the academic year 1964-1965, during part of the summer of 1965, and on Saturdays throughout the academic year 1965-1966. They played a major role in shifting the Project away from "Curriculum ϵ " and "Curriculum α," and toward "Curriculum β." They taught the Project that the teacher should not stand and thereby dominate the room, but rather sit and work with students as equals; that the teacher should only occasionally address the entire class, but usually talk "privately" with two or three children at a time; that children should sit at tables in groups of three or four, working together, but not necessarily working on the same tasks that other groups are working on; and that much of what the children do should involve the manipulation of physical objects more mathematically suggestive than a pen or a pencil or a piece of chalk.

40. (unnumbered). Lesson taught on a Saturday, during the 1965-1966 academic year.

At this time, these children were mainly in grade 6, though some were in grade 7.

Content: Discussion of "Which is bigger, a jackknife or a piece of paper."

This question occurs in the *Discovery* book, in Chapter 42, where it is intended to get children to realize that quite general attributes like "bigness" can be interpreted more specifically in a wide variety of ways: "big" in terms of *weight*, or in terms of *mass*, or in terms of *girth*, or in terms of longest linear dimension, or in terms of how large a door would have to be for it to pass through, or in terms of how large a box would have to be in order to contain it, or in terms of how much water it would displace if you immerse it, and so on. However, in 1960 we would have dealt with this question as a discussion question, and the teacher would have stood at the front of the room and *talked* with the children about it. This procedure *can* be made to work. Yet in 1967 we would probably not use it -- and, indeed, in the lesson recorded here we *did not* use it. We handed the children in the group -- that is to say, three of them -- a jackknife and a piece of paper, and asked them to decide which was larger. The results of such use of physical materials have been very enlightening, both for the teachers and for the children.

Appendix D

Questions of Goals, Philosophy, Etc.

The careful reader of this report will have noticed how often the Project has felt itself coming into conflict with an invisible restraining wall that impeded progress. After about a decade of protracted discussions and consideration, the Project believes this obstacle can be correctly identified as -- to borrow John Kenneth Galbraith's useful phrase -- the "conventional wisdom," this time the conventional wisdom of educational theory rather than the conventional wisdom of economic thought.

This appears to be a real issue, and not an imaginary one. The rigidity -- or, as J. Richard Suchman perhaps more accurately labels it, the "homeostatic propensity" -- of educational institutions is nowadays alarming. Inadequacies of schools are conspicuous wherever one looks -- so, too, are possible roads to improvement. What, then, impedes progress?

After considerable experience, the Project would identify among the leading obstacles the following:

i) *The (often unconscious) assumption that "measurement" and "science" must be central to all attempts at improvement.* This assumption cannot withstand even moderately close scrutiny. Surely many of the things we most desire to accomplish by education cannot, at present, be measured. Yet we must continue to concern ourselves with them. The very *quality* of life itself in the United States today cannot be measured in sensible terms, and we have often made unwise decisions when we have employed oversimplified or misleading measures.

ii) *The (again, often unconscious) assumption that we can always state our goals explicitly.* Even within the past week the Project has been involved in advising on the architectural and programmatic design of a new school. Halfway through the meeting the conventional wisdom seized a group of participants who thereupon formed a committee to "state goals"; their report added nothing to anyone's understanding of the task at hand. Doubtless there *are* times when some clarification of goals is exceedingly helpful; but there are many times -- and possibly many *more* times -- when goals must be left implicit and temporarily unexamined. We often discover our destination when we reach it.

iii) Presumptions of excessive control. Physicians operate as autonomous professionals; education ordinarily assumes that teachers cannot. The Project has collected thousands of instances where someone's desire to maintain control has impeded progress. Teacher initiative has been discouraged; quite valid student initiative has been discouraged; parental interest has been discouraged; diversity has been discouraged; innovation has been discouraged; and even superior quality of performance has been discouraged -- in every instance because such developments were perceived (probably correctly) as threats to someone's control. A superintendent cannot "control" a superior school system, although he can surely control a mediocre one.

iv) The non-existence of change-producing agencies. This hardly requires comment.

v) Domination of curriculum and instruction by commercial textbook series. The harmful effects of textbooks have been reported by various authors, including Peter Schrag, Hillel Black, and John Goodlad.

vi) The assumption that communication is co-extensive with verbal communication. Here, again, the assumption is usually implicit, rather than explicit. It is not the less damaging for being unnoticed. In fact, whether one one is dealing with the child learning mathematics, the teacher learning the art of teaching, or the educational specialist studying curriculum and instruction, spoken and written sentences in English (and in mathematical notation) by no means suffice to carry the entire burden of the communication.

Learning and communication are multi-sensory processes, and far more heed must be paid to this in the future.

vii) The tendency to base educational decisions on written reports. This is a corollary to the assumptions of verbal language, explicit terms, and a high degree of control. Anyone who has lived closely with schools at the classroom level knows how badly the classroom realities are usually reflected in written reports, and how harmful to classroom experiences are many of the decisions that are based upon such reports.

The Project has attempted to articulate its views in a number of articles and pamphlets (namely, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, and 14 on the list below); similar or relevant discussions appear in the writings of many other authors. Here is a partial list of references:

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These references deserve study. We present a brief indication of their rather elusive message by referring again to the Robert Schaefer article cited in Appendix B, and comparing it with some remarks by Erich Kahler on the nature of art. Perhaps our underlying theme is that "science" -- indeed, every appropriate rational approach -- is valuable and urgently needed, but that "curriculum innovation" has not prospered in recent years as it should have done because the conventional wisdom has failed to recognize the important role of the artist-practitioner, and has not enabled (or even permitted) most classroom teachers to play any important role as artist-practitioners in innovation in curriculum and instruction. Man made beer for many centuries via the skill of the brew-master, before any "science" was available to come to his aid. Had brewmasters been somehow constrained to be "scientific," we should likely never have developed a beer that civilized people would wish to drink. We do not now have anything like an adequate repertoire of school experiences that teachers and children can reasonably wish to share. Indeed, the barrenness of our classrooms is awe inspiring. Instead of dreary textbooks we could have exciting classroom experiences. Enough examples already exist to suggest rather clearly how much more is possible.

In the article cited earlier, Dean Schaefer writes:

> The primary job of the school is to teach -- to provide instruction in the various skills and subjects deemed crucial for the young. Society has not expected the school to be systematically reflective about its work -- to serve as a center of inquiry into teaching -- for the simple reason that there has seemed nothing of great complexity in the instructional task, few problems in teaching which demand serious investigation. Educational reform, therefore, has historically focused upon modifying the curriculum or raising the standards for admission into teaching.
>
> The truth, however, is that we can no longer afford to conceive of the schools simply as distribution centers for dispensing cultural orientations, information, and knowledge developed by other social units. The complexities of teaching and learning in formal classrooms have become so formidable and the intellectual demands upon the system so enormous that the school must be much more than a place of instruction. It must also be a center of inquiry -- a producer as well as a transmitter of knowledge.
>
> By a school organized as a center of inquiry, I imply an institution characterized by a pervasive search for meaning and rationality in its work.

Fundamentally, such a school requires that teachers be freed to inquire into the nature of what and how they are teaching. Discovering new knowledge about the instructional process is the distinctive contribution which the lower schools might possibly be expected to provide. As every teacher knows, however, pedagogical strategies cannot be meaningfully separated from content, and there also must be continuing opportunity for the teacher to inquire into the substance of what is being taught. Finally, no school can be reflective about its work or serious in its commitment to learning if students are not similarly encouraged to seek rational purpose in their own studies.

It is wholly within our command to make schools more intellectually exciting institutions -- places not only where youngsters are pressured to learn a little of what is known, but also where adults investigate matters not yet understood. All that might be required to create schools which serve as centers of inquiry may be beyond our present ability to specify, but the main outlines seem clear enough.

...

The university professor ordinarily carries a formal teaching load of six to nine hours per week, and he complains that too large a fraction of his remaining time is consumed in preparing for the ordeal. Why do deans and university trustees hold such markedly different expectations than those held by superintendents and boards of education? An important part of the answer, of course, is that professors presumably devote major energies to original research in a particular field, but it is also relevant that a different conception of the teaching task obtains. It is assumed, and not simply piously, that a college course must be illuminated by a scholarship which ranges far beyond the limits of any set of texts or outside readings and by a continuous reexamination of appropriate sources and interpretations. If society were to take seriously the job of teaching in the lower schools and, particularly, if teachers were to be encouraged to inquire into the substance of what they are teaching, or into the nature of the students with whom they work, or into the learning process itself, it is apparent that a teaching load of more than twelve to fifteen hours per week could not not be condoned.

. . .

In American secondary schools the dreadful legacy of the Carnegie unit and the dreary persistence of the assign-study-recite method of instruction inhibit intellectual pleasure for both teachers and students. Both notions preclude the development of the school as a center of inquiry. If pupils are to inquire into the substance of what they study, we "have to remember," with Whitehead, "that the valuable intellectual development is self-development."[105]

What is the mysterious difference between the senior in high school and the freshman in college, that the latter may stretch his mind in libraries, museums, and laboratories as well as when he listens to instructors? If it be feared that many high school students are not sufficiently disciplined for self-directed learning, then surely we should recognize that some schools may require other types of personnel in addition to teachers. Certainly, however, teachers need not be forever confused with supervisors of study halls or petty officials in places of incarceration.

In the elementary school the mystique of the self-contained, and therefore unrelieved, classroom has excluded teachers, during their working hours at least, from the adult world. Such deprivation may meet some psychological need of children--although I gravely doubt it -- but it has most assuredly prevented teachers from systematically inquiring into the rationality of their craft.

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[105] Dean Schaefer's reference is to: Alfred North Whitehead, The Aims of Education (New York: The New American Library, 1929), p. 13.

The system of executive authority which characterizes American education -- hierarchical flow from the top down, from the superintendent's office through the supervisory staff to the worker-teacher -- has deep roots in our history. I am strongly convinced, however, that present circumstances have drastically altered the need for hierarchical control. The number of teachers who do not hold a bachelor's degree grows smaller each year. The quality of the initial preparation in the teaching field steadily improves. A great many teachers are far better educated -- particularly in their teaching specialty but often also in their general liberal preparation -- than the supervisors under whose "guidance" they presumably work. Teaching is increasingly perceived as affording a lifelong rather than a floating career. Most importantly, teaching is now attracting an ever-larger fraction of exceptionally able and well-motivated young people. Some of our best college graduates seek positions, not in industry, but, in their own terms, where the action is. If it isn't in education, it could be.

Under present circumstances, vigorous, alive, intelligent, and socially committed young people often find the schools lonely and intellectually barren places. The social norm which prevails is to treat one's fellow teachers, new or experienced, in a friendly but nonintervening manner. There are few opportunities for serious discussion, and the lack of a developed, specialized vocabulary and meaningful sets of pedagogical concepts makes the professional communication which does occur nebulous and imprecise. Teacher-education programs rarely prepare teachers for powerful and continuous professional association, but ordinarily aspire only to readying the neophyte for the here-and-now demands of the job.

For really able and dedicated young people this physical and intellectual isolation can be intolerable. The need for productive colleagueship becomes especially acute when one realizes how much exciting work could be tackled. Studies of particular pupil populations, production of specialized curricular materials, the development of particular pedagogical strategies, experimental efforts to translate unduly complex content into terms appropriate to an elementary or secondary school youngster, and the development of non-verbal approaches to learning are only suggestive of the range of activities in which teachers might be engaged.

...

Behavioral science, of course, has no monopoly on the function of providing fresh perspectives. There are many modes of analysis which can be applied to the classroom. ... On several occasions John Fischer has suggested the development of a criticism of teaching comparable to established traditions in literature and the arts -- an aesthetics of the art of teaching, so to speak.[106]

...

Although other pedagogical issues may be involved, dramatic failures in teaching are often due to the discontinuity between content organized for the demands of collegiate specialization and content appropriate to the general education of younger pupils. Translating the patterned abstractions and conventions of upper-level college courses into terms appropriate to an elementary- or secondary-school program requires enormous effort and a considerable intelligence. The difficulties in such translation undoubtedly explain our tendency to shunt academically unresponsive youngsters into shop courses, remedial work in general mathematics, or, in cliché travesty, into Fly-Casting I and II. The difficulties do not excuse us, however.

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[106] Dean Schaefer's reference here is to: John H. Fischer, "The Prospect of Teacher Education," in The Journal of Teacher Education, Vol. XVI, No. 4 (December, 1965), pp. 381-398; or the Annual Report Issue, 1963-1964, of the Teachers College Bulletin, pp. 1-9.

I suspect, also, that our failure to stimulate interest in the organization of substantive content for instructional purposes explains the disillusionment of many academically able students who elect to teach in the lower schools. Seeking to perpetuate his own intellectual pleasure in exploring a particular subject, an unwary beginner may imagine that pupils will almost automatically come to share his joys and enthusiasms. The disappointments of the first year for such a neophyte may literally drive him from the classroom. Unless he can direct some of his intellectual energies to the reformulation of materials from his teaching field, he is likely to be punished rather than rewarded by the responses of his charges. Teacher-education programs, I am afraid, have been disastrously unsuccessful in convincing prospective teachers of the potential intellectual rewards in the manipulation of content for particular audiences.

Who is to help the new teacher to communicate the excitement and relevance of his teaching field to pupils whose attitudes and social experience may be ill-equipped to respond to either the discipline or the pleasure of systematic study? Obviously, responsibility cannot be delegated to the substantive collegiate departments, for they are neither staffed nor inclined to take on the task. A possible answer is Conant's clinical professor of education, but master teachers of the sort Conant envisages can only introduce the apprentice to the problems he will face as a regularly employed teacher in another school. Presumably, the responsibility now is upon the professor of education, but scholars interested in methods of teaching are normally more concerned with other aspects of practice than the organization of content for particular classes. This is as it should be, for it is impossible for a single professor to anticipate the range and variety of pupil populations with which beginning teachers will be confronted. But the basic fact remains, unfortunately, that there are few specialists in schools or colleges who know anything worth teaching about how to organize history, literature, or biology for particular groups of pupils. I believe that groups of practicing teachers, working in collegial association in individual schools or at least in individual school systems, must inquire into the problem.

Let me be very clear on this point. I am definitely not suggesting that individual schools make such loose and fanciful translations as to lose sight of the original text. If we have learned anything from the new curriculum movement, it is that every child, no matter what his measured capacity to learn, has the right to direct scholarly experience in the sciences, the social sciences, and the humanities. But every child's experience of scholarly inquiry poses new questions and new problems for the teacher. No matter what the level of detail of curricular packages, their full value can never be reached without the the intervention, the thoughtful translation if you will, of a wise teacher. Even if it were possible for external curriculum writers to do the full job, I would resent the teacher's exclusion from his best chance of intellectual delight. Intelligently conceived curriculum materials, of course, need not attempt to impose a rigid uniformity but can deliberately build in modes and possibilities for departure by the individual teacher. And, finally, as I have earlier asserted,

> how can children fully know the dynamism of
> learning if the adults around them stand still?[107]

Compare the creative role in curriculum and instruction which Dean Schaefer envisions for the teacher as artist-practitioner, with some of Erich Kahler's remarks on the nature of art:

> As an activity, art borders on other human activities with which it has, in part or as a whole, certain traits in common: craftwork, science, philosophy, historiography. Painting, sculpture, architecture are connected with handicraft (indeed technology), and even music and literary work of all kinds are partly constituted by techniques. With science, art has in common the exploratory character of its work, the intent of ever expanding cognition; with philosophy, its conceptual nature, its concern with ideas; with historiography, its descriptive element. Art differs, however, from every one of these activities by specific variances of mode of these common properties, by its special combination of properties, and by additional properties which are exclusively its own.
>
> Everybody will agree that the aim and activity of science consists in the acquisition of ever wider and deeper knowledge of the nature of reality. The same can be said of philosophy. And it is no less true of art, only here this is not so easily recognizable because another property of art tends to conceal it.
>
> ...
>
> Art too is involved in the exploration of reality, the penetration of an ever wider and deeper range

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[107] From: The School as a Center of Inquiry by Robert J. Schaefer. Reprinted by permission of the publisher, Harper and Row.

and increased complexity of reality, and in this respect it has a demonstrable evolution, just as science has. This is an evolution of the nature and scope of reality rendered, and implicitly of the forms of rendering, the modes of expression. A newly detected reality involves new forms and techniques of expression. "Content" and "form," as it is well established by now, are but two aspects of one and the same thing. New "contents" call for new "forms" of presentation; they are not expressible, they simply do not exist without their new appropriate form. And just as science, through its new findings, broadens and changes the picture of our reality and in this way, as well as through the application of its findings, changes our reality itself, so art, through its reaching into new complexities and levels of reality, extends the scope, and changes the nature, of our consciousness, and by this means of our reality itself.

In science the crucial importance of novelty is quite evident. What else does science seek than additional knowledge, better knowledge, and that is new fact-finding, new knowledge. As far as art is concerned, however, scarcely any explicit attention has been given to the equally essential role which the search for the new plays in it. Here, the search for the new, the impulse toward the new, means even more than additional material, increased profundity and complexity. It implies other properties which we, more or less consciously, sense and admire in any "paradigm" work of art, such as freshness, vitality, vigor, authenticity, expressiveness, precision, truth, and even emotional impact. All such properties and, as we shall see, even more are inherent in that one characteristic of art: its thrusting out into the sphere of the unknown, the hitherto unachieved, "the ground that never a word has trod," as Rilke put it, and we may add, "never a stroke of the brush, or a conventional sound." For it is this effort to express something heretofore inexpressible, to grasp and to shape something for the first time, it is this "for the first time" that gives a work of art its lasting freshness and vitality, its genuineness of language, its convincing vigor, so that ancient works, whose scopes and styles are by now utterly familiar to us and in one way or another left behind by the endeavors of our age, are still fully alive, and we are able to enjoy them as if they were created today. The trace of that ultimate effort that created them persists in them, the longing, the struggle, the suffering, the immediacy of all primal creation. When we feel certain works to be of secondary quality it is because all this is lacking in them; they echo, iterate and imitate the achievements of masters.

It is this drive toward the conquest of the unknown and inexpressible which also makes for the evolutionary and sometimes revolutionary character of artistic activity; which causes the expansion of scope, the growth of complexity, the development of new dimensions, forms and techniques, traceable throughout the history of the arts.

There is a further most important property of art, which derives from the same source. By penetrating into new spheres and dimensions of experience, developing new forms, creating new reality, artists like Masaccio, Giotto, Leonardo, Titian, Rembrandt, the impressionists, Cézanne, Van Gogh, Picasso, or Dante, Shakespeare, Cervantes, Goethe, the romanticists and the symbolists, Flaubert, Proust, Joyce, Kafka, or Monteverdi, Bach, Mozart, Beethoven, Wagner, Stravinsky, Schönberg -- to mention only a few of the most decisive figures -- such artists did not just indulge in an arbitrary play of their imagination; they were pushed along or guided in a certain direction by the human condition, the stage of perception and experience, and implicitly the techniques of expression, of their specific period; and the new experience of reality, indeed the new reality itself they reached was a reality in the making that lay hidden under the conventions of the epoch. They presented a human condition, they liberated forms and experiences that were to become the reality of the next age. That means: whatever they presented was not just this or that individual story, portrait, scenery, phenomenal structure, or combination of sounds; it reflected a human condition in transition. When we admire in a portrait by Titian, Dürer, or Holbein the deep physiognomical grasp of an individual personality, what strikes us is not only the visual elucidation of this specific character, but implicitly the close, meticulous and yet synoptical accuracy of rendering as such, which these Renaissance masters have achieved. Here, likeness represents the last frontier of visual exploration. Again, when we turn from these portraits to portraits by, say, Van Gogh or Kokoschka, we are confronted not merely with a peculiarly sharpened and intensified characterization of the specific person depicted, but with a new structural and psychic depth of appearance which has been revealed by these artists. New phenomenal vistas have been opened up, uncovering the new reality of our age. The same applies to the progression from a scene or scenery presented by a Renaissance painter to one by Rembrandt, by Claude Lorrain, and by Constable, and further on one by Turner, by Monet, and by Cézanne. What a modern "non-objective" painting expresses in its line-and-color construction is not just this particular formal combination, but implicitly phenomenal relationships of a more general order. In literature we need only trace the corresponding development from the baroque novel to those of Flaubert, Proust, and Joyce; in music the steps from Haydn to Beethoven, to Wagner, to Stravinsky and Schönberg. By virtue of their frontier character, of their acts of revelation, these works bear a generally human significance. Their presenting in, and effecting through a singular selective unit a coherency of the broadest import establishes their symbolic quality. Thus the special creativity of art implies an inherently symbolic quality.

It is this symbolic quality which distinguishes art from science. Both activities have in common the search for the nature of reality, their disclosure of new reality. But the manner of this search and the dimension of reality it reaches are characteristically different in one and the other activity. While science deals with factual reality and approaches it in a direct, immediate way, art proceeds in a symbolic or metaphoric fashion. It presents single individual entities carrying a general purport. The abstraction and generalization of science is extrinsic, it seeks to arrive at strict or statistical laws which call for mathematical expression; the abstraction and generalization of art is intrinsic, it shows macrocosmic through microcosmic coherences, and its form of expression is therefore symbolically or metaphorically variable. It is in the nature of this kind of endeavor, which is aiming at the presentation of a whole, that each work of art is complete in itself, or supposed to be complete in itself, and that it is, accordingly, an individual work. This characteristic of art, that each of its works is complete in itself, has blurred the recognition of the fact that art also has a definite evolution.

...

In the present investigation I have attempted to show that an approximately reliable recognition of the nature of art is very well possible when we consider art as a human activity characteristically different from other human activities. We have found that the special creativity of art consists in the discovery, and that means implicitly creation of new reality, reality that has not been raised into our consciousness before, that expressing something for the first time is a crucial feature of art which appears to be the primary source of other qualities, less strictly identifiable and rather felt than clearly recognized, such as vitality, authenticity, precision, truth, etc.; that art reaches the new reality in a suprarational, visional, metaphoric way, and that accordingly the reality it presents is a microcosmic whole reflecting a macrocosmic whole -- which establishes the symbolic quality of art; that this very same manner of proceeding and presentation calls for another crucial property of art: its aiming at organic wholeness, integration, "harmonization" of an ever more complex and discordant reality. In those works of art which, by venturing into new spheres and dimensions of reality, break down worn, conventional harmonies, the emphasis is on intensity, vigor and accuracy of expression. Such works are the pioneers, preparing the path for other works which attempt to rally these gains in a new, more broadly balanced whole and to achieve completion and perfection on a larger scale. The correspondences and convergences of parts, elements, and symbols which constitute such a work of art are amenable to accurate demonstration, which means a, however limited, verification of art.

...

...this much is incontestible, that while scholarly work is always incomplete, always in progress, always open to being expanded, corrected, indeed supplanted by subsequent knowledge, the work of art is forever contained and complete in itself, always sufficient in and unto itself. Even though in one period or another its effect may be greater or less, and in any case different, nothing that is done later can take away its validity, can touch its artistic quality, which is the quality of a consciously created whole, a unity, manifold in its aliveness, multiple in its relations and interrelations. With respect to this fundamental characteristic of art there exists, then, no difference of time and place, and to this degree certainly the criterion of art is independent of historical development and cultural milieu.

But artistic quality has another ingredient, one that I hold to be quite as essential to it, and this, precisely, has to do with historical development, with the development of human consciousness and outlook, with which art is in the highest degree involved.

Art is one of man's forms of expression: what is expressed -- consciously or unconsciously -- is the condition humaine as of a given moment; how it is expressed is the result of an ineluctably personal mode of grasping that condition. What is seen and how it is seen cannot be separated, but are one and the same. Art's statement does not simply consist, as is so often thought, in an especially complete, especially successful reproduction of something that has already been seen, but rather in the creation of something not yet generally, not yet clearly seen, something that is only drawing near, potential, pre-existent in the undercurrent of the period, and that art brings to actuality, to existence. This "creation" of an existent out of the latent -- precisely this is the creative process; and the very things that often appear to be arbitrary fantasy or mere subjective representation will prove a thrust of discovery.

All great art brings a new reality into being, a new world that it teaches us to see. It widens and deepens the space of our vision, our consciousness. Certainly this is not always accomplished by a single powerful advance -- as in the case of Nicolò Pisano, Giotto, the Van Eycks, Breughel, Rembrandt, Cézanne; often it is done by many gradual steps and variations, many different flashes of insight on the part of many different artists, very personal conceptions that only later, in retrospect, become recognizable as having contributed to a total transformation of our world. Among artists there are those who are the explorers, the experimenters, the conquerors, who make the decisive breakthroughs; there are others who occupy the territory that has been won, who cultivate it, make it fruitful, who work out in an infinite variety of detail what has as yet been only broadly

> conceived. But in them too the element of innovation, of transformation of seeing, transformation of reality, is an integral constituent of their artistry and of the artistic quality of their works.
>
> But what is it that gives life to a work of art, what makes the difference between it and some conventional product, technically excellent as it may be? Simply that immediacy of perception which is only possible when something is seen for the first time. The man who merely follows or exploits tradition is repeating in a superficial degenerated form what the masters before him had exerted their utmost effort and inmost potentiality to generate. Repetition necessarily becomes mechanical, lazy, shallow, tedious, falls behind what had formerly been initial, origin-al. The spurt of life is always happening for the first time. The creative impulse acts right at the forefront of the inexpressible, to wrench form out of what has hitherto been formless. Where tradition works upon him, the true artist extends and transforms it. And a work where we feel that pulse, that impulse toward the initial and initiatory, with its self-surpassing labor, its self-surrender and self-forgetfulness, is the only work that can still move us in a world that has far outgrown the one from which it emerged.[108]

Is it not possible to read Kahler's words and to think, not of Picasso or Beethoven, but of a creative teacher developing appropriate classroom experiences for a group of students? Is teaching unreachable by a suitably-developed rhetoric of artistic criticism? Very little improvement can be seen in most classrooms as a result of past attempts to tie the tail of teaching to the kite of science. No one would deny the increasing value of the behavioral sciences -- but it is perhaps at least as unwise to deny the possibility of approaching teaching, curriculum, and instruction, as a single unity, an undivided activity, to be regarded as an art.

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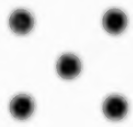
[108] From: Out of the Labyrinth: Essays in Clarification by Erich Kahler (57), pp. 99-112. Reprinted by permission of the publisher, The Macmillan Company.

This becomes all the more significant when one realizes that communication today is not limited to writing; we have magnificent vehicles available in motion picture film and video tape. A classroom experience, once created, is thereby enabled to assume a new existence, and a permanence comparable to man's other works.

Appendix E

An Example of Some Lessons in Mathematics for the Primary Grades.

As discussed in the body of this report, the Madison Project introduction of mathematics in the primary grades is based upon *the actual operation of counting* (in a quite traditional sense), and upon children's recognition of *visual patterns* such as



Beyond this, it builds upon sharing out discrete objects, bringing them together, taking some away, etc. We believe these -- and not the incantation of the word "set," which is useless at this stage -- is the child's normal road of entry into mathematical ideas.

In order to give some idea of how these lessons proceed, we reproduce here some notes by Beryl S. Cochran. Mrs. Cochran is a classroom teacher, rather than a mathematics educator or mathematician, and she writes in the present-day idiom of classroom teachers. Her language is reproduced here in the original form of her "Notes."

It should be mentioned, in passing, that Mrs. Cochran's notes relate to one out of more than four nursery school and primary grades programs the Project has developed or is now developing. The earliest, developed by Mrs. Cochran, R. B. Davis, Katherine Vaughn, and others, was mainly a downward extension of the Project's previous program for grades four through eight. A second program, *by far the most sophisticated*, was developed by Doris Diamant Machtinger, and is described in four reports, namely:

Robert B. Davis, *A Modern Mathematics Program As It Pertains to the Interrelationship of Mathematical Content, Teaching Methods and Classroom Atmosphere (The Madison Project)*, 1965. (Report submitted to U.S.O.E., and available from The Madison Project), pp. 11-12.

Doris Diamant Machtinger, *Experimental Course Report/ Kindergarten (Report No. 2, July, 1965)*, available from The Madison Project.

and in two other preliminary reports by Mrs. Machtinger that are available from The Madison Project, 8356 Big Bend Blvd., Webster Groves, Missouri 63119.

Mrs. Cochran's program, the flavor of which is suggested by her notes, is chronologically the third program of those being considered (various Webster College students, and especially Mrs. Joan O'Connell Barrett, having developed several others that we do not list here), and is the "simplified curriculum for primary grades" that is referred to elsewhere in this report as "Curriculum γ".

Finally, the Project is presently embarked on a fourth primary grades program that is planned to follow a child's natural tendency to think quantitatively -- indeed, to follow this so closely that the program has almost no "separate" existence of its own. Several English educators -- notably Leonard Sealey, Edith Biggs, and Geoffrey Matthews -- are far ahead of the Madison Project in their ability to do this successfully at the present time.

We give one example:

Alexandrea (a girl, aged five years, four months), to her father: "Daddy, how old is Uncle Eddy?"

Father: "He's three years younger than I am."

Alexandrea (who knew her father was "41," but could not really comprehend what "41" meant -- although she repeatedly tried new attacks on the problem): "Daddy, count backward from 41 to see how old Uncle Eddy is."

There is obviously the foundation here for an exceedingly "natural" approach to learning mathematics, provided methods can be found to let children explore such matters as easily as they explore their own physical environment.[109]

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[109] One instance of this kind of thing is suggested by a letter received today (October 23, 1967) from a teacher in New York City, Miss Helene Silverman of P. S. 51, Queens. This letter is reproduced at the end of Appendix E.

This fourth program is, at present, more an idea than a reality. Mrs. Cochran's program is at present quite real and very much in use. Here are her notes in their original form:

A Collection of Written Material

to be used with

Primary Teachers

in the

Madison Project Teacher-Training Workshops

in New Curriculum Mathematics

Summer, 1967

by Beryl S. Cochran

Mrs. Cochran wishes to express her sincere thanks for the cooperation of the administrators, teachers, and children of the following schools:

The Special Educational Program conducted by the Junior League of Bridgeport in the interest of Integrated Education

Chicago Public Schools, Chicago, Illinois

Hampton Institute, Hampton, Virginia

Kingsville Public Schools, Kingsville, Texas

Los Angeles Public Schools, Los Angeles, California

Sierra Leone, East Africa

Weston Public Schools, Weston, Connecticut

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NOTES FROM A MATH SPECIALIST

A First Lesson in Arithmetic
With
A First Grade in South Texas

Notes from a Math Specialist
Beryl S. Cochran
October 1965

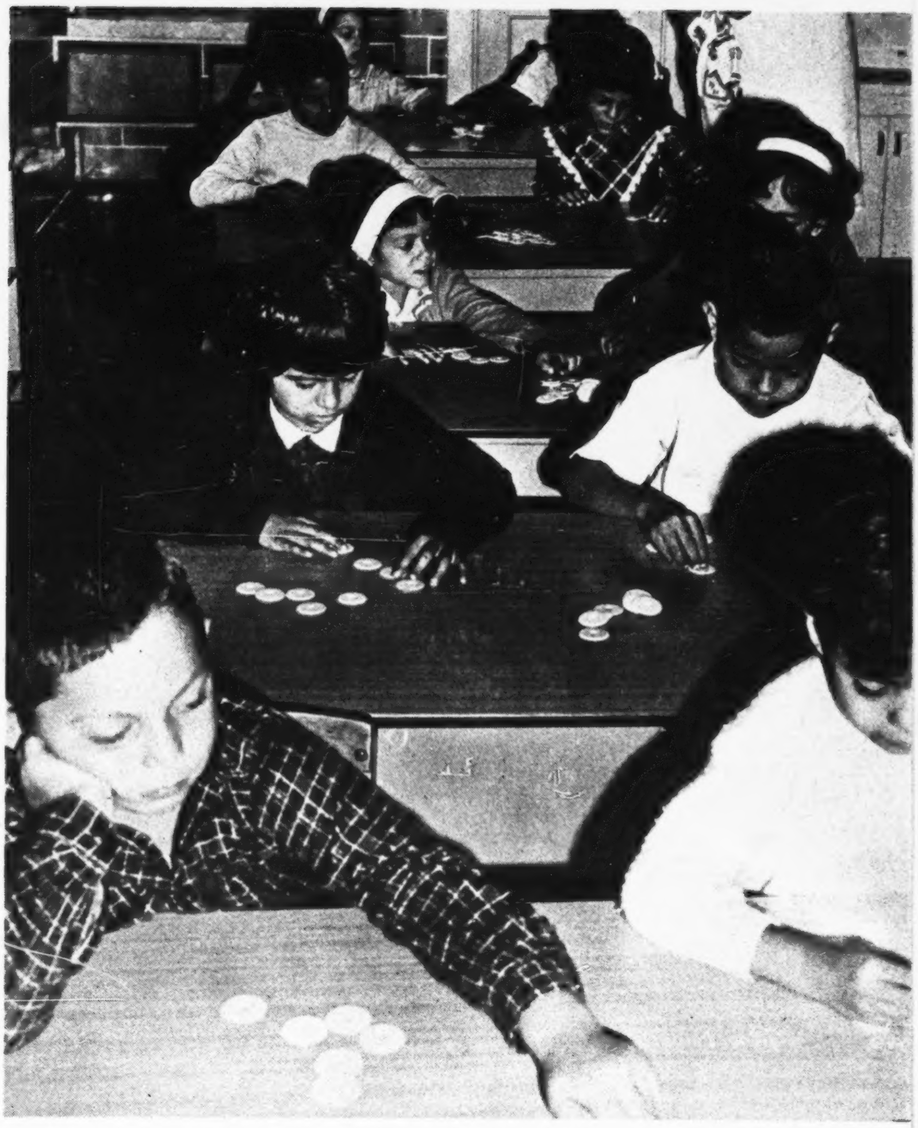
One sunny morning in September, I walked into a classroom of twenty-eight bright-eyed Latin American children in South Texas. They had come to school for the first time five weeks ago, and most of them had had very little exposure to the English language. Their classroom work so far this year had been geared to the language arts program, and the children had certainly responded beautifully, for I was able to teach this class their first arithmetic lesson in English.

The children were sitting two to a small table, thus giving them an ample surface on which to work and share. I had taken with me a draw string bag which contained round plastic discs with holes in the center. These discs resembled the common household washers. They are a simple counter which children seem to like.

I put a handful of these washers on each table and said:

"I'd like you all to have about ten washers."

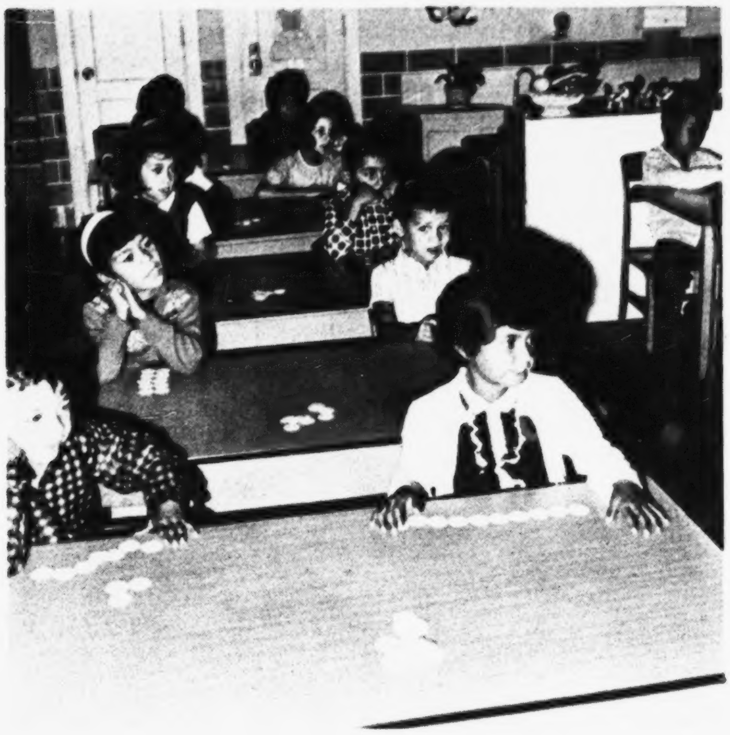
Each of twenty-five children pulled TEN washers in front of him --
NO MORE - NO LESS.



Three children did not. One of these children pulled six washers out of the pile with one hand and just let them sit there. Another pulled eight, and then she started arranging them in various patterns. The third took a few washers from the pile and just held her hand over them, looking around to see what the other children were doing.

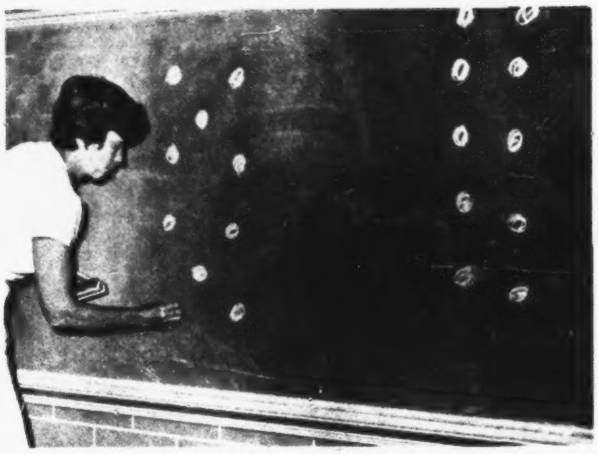
The three children who didn't have the correct number TEN in front of them perhaps couldn't count as well as the others or perhaps didn't understand my directions, which, of course had been given in English. I watched these three children carefully during the remainder of the lesson to see if it was the number concept, the mechanics of counting, or my English-language directions that kept them from fully participating.

As I walked to the front of the class, I said, "I see many of you have grouped your washers."



"Some of you did it like this, and some like this."

I drew diagrams on the board showing patterns that various children had made with their ten washers.



Since the washers were in patterns on their desks, I could more or less observe their thinking. There were children in the class who obviously were able to count. Some used the "one, two, three,..." procedure; others counted by letting their eyes scan the washers, which made patterns that were easy to count at a glance. Some students perhaps were thinking in terms of the addition operation: five plus five equals ten.

The classroom teacher helped the three students who hadn't counted out their ten accurately. And they seemed to understand the problem at hand with this little extra help.

I asked the children to push all their washers into one pile in the center of their tables.

I now thought I'd see if the children could recognize a numeral.

As I wrote a large six on the board, I said,

6

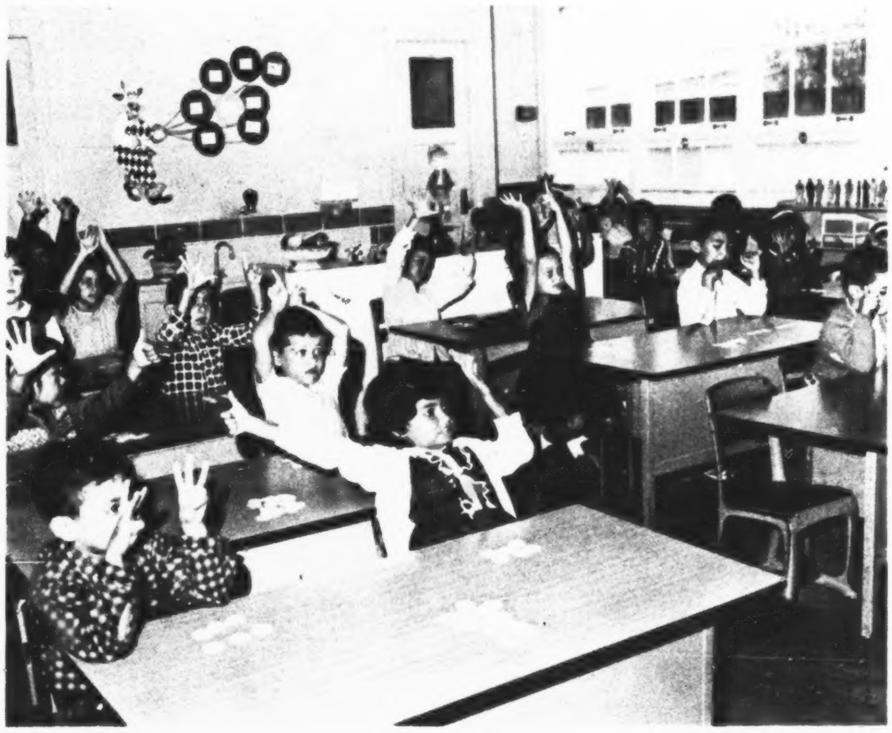
"Take this number —
What did I write on the board."

The children told me, "SIX."

I then asked, "Did you all take six washers?"

The children replied in unison, "YES."

I thought I'd check their knowledge of six in another way.



"Show me six with your fingers."

From this picture, I think the reader can see that there was a diversification of understanding. The boy at the second table on the left understood very little English. However, he did have six washers in front of him. I doubt that he understood my statement, "Show me six with your fingers." The boy at the far right is one who had had difficulty with the counting out of TEN.

I had some children come to the board to write six, while the rest of us wrote it in the air.

I had written the numeral six. I had had the children count their washers. They had shown me six with thei fingers. I had diagrammed the patterns, and we had written six in the air and o the board. And they all ended up with six washers front of them.

I now asked, "How many do you want to take?"
Someone in the class said, "TEN."
"All right, take ten." And I wrote ten on the board.
I then asked, "What shall we do now?"
And to my surprise a little boy said, "TAKE BACK SOME."
"All right, take away two. This says ten minus two equals."
As I wrote this on the board I was very careful to make the minus sign heavy and dark.

"How many washers are left?----
What should go here?"
Maria said, "EIGHT."
"I guess you are right;
I'll put eight here."
"Do you all have eight?
"Let's count."

So the lesson went. The children could count, group, and recognize numerals. I was able to ask them questions that gave them a participating part in the direction of the lesson.

At one time I said, "José, how many do you want to take?"

"FIVE." I wrote five on the board. 5

"Rose, how many shall we take for you?"

"SEVEN." I wrote plus seven. $5 + 7$

"How many have we in all?"

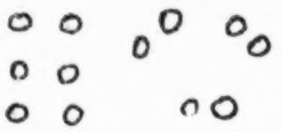
"TWELVE." I wrote equals twelve. $5 + 7 = 12$

"How many shall we put back, Ron?"

"TWELVE." I wrote twelve minus twelve. $12 - 12 = 0$

We were also able to do such problems as:

Take two, three times



and I diagrammed the various ways the children did this grouping.

We even took twelve washers and divided them into two equal groups, as if we were going to share the group of twelve with our mothers; and of course, we had a pile of six for ourselves and a pile of six for our mother. $12 \div 2 = 6$

Juanita observed that if we wanted to share our twelve washers with our father, too, we wouldn't get so many for ourselves. We would only get four, she said, and when we worked it out we found this to be true. $12 \div 3 = 4$

The teacher and I walked around, observing the children as they worked with their washers. We could help a child now and again, but they seemed to feel pretty much in control of their own learning.

We constantly counted, I drew diagrams of their washer patterns on the board, and we carried on a very low, easy conversation concerning the operation involved. I could almost see what they were thinking and learning through this whole lesson by what they were doing.

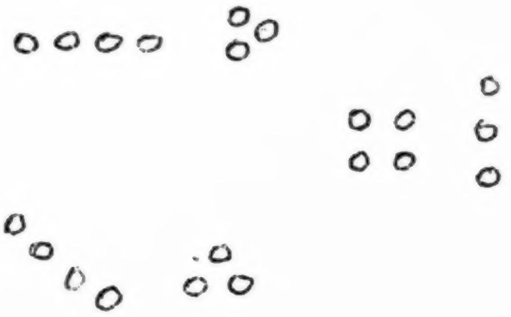
Before the lesson was over, I wrote on the board "four plus three" and asked the children to do this mathematical sentence. No one read this number story out. The children did it by recognition of the numerals and operation sign, and even the three whom I had thought couldn't count at the beginning of the lesson grouped their washers -- four washers and then three washers.

4 + 3

We all counted one, two, three, four

plus

one, two, three.



"How many have we in all?"

The children told me, "SEVEN."

4 + 3 = 7

"Who wants to write seven here?" I asked.

Several children volunteered. I asked José to come write it for us on the board while others wrote on their desks with their fingers. I felt the three "non-counters" were now both counting and grouping. Many of the children were handling the operations, but, since their handwriting and numeral formation was poor, they were unable to put this thinking on paper. Yet the children were thinking and analyzing mathematical sentences.

Why was it that these children were so responsive and seemed to understand almost everything I said or did? They obviously were alert, inquisitive children, ready and able to learn. Was this due to the fact that they came from a verbal culture?

I was told by some observers at the completion of the lesson that I had used mathematics as a medium for communication, and thus I was able to present a lesson similar in content to any I've taught in suburban Connecticut. However, I think there was more to it than this. Mathematics can be used as a medium of communication, but it does take two to communicate, and this lesson seemed to be a dialogue between teacher and class.

I have had to stop and think why this class was so good. Why did these children respond so well? I have analyzed it somewhat as follows:

1) These children came from homes where there is family life and real communication, even though their communicating language is not English.

2) No doubt these children have been to the store for their parents and have played hopscotch and baseball in vacant lots with their friends. They have had things taken away from them, thus experiencing subtraction. They have saved bottle tops, stones, and shells, experiencing addition and have shared objects, thus experiencing division. That is to say, they had had many pre-school mathematical experiences, even though their thinking was in a foreign language. They had been thinking and analyzing much as the child in suburban Connecticut would have done in pre-school years.

3) The children were respected as learners. It was obvious that the principal was proud of his school, his teachers, his children, and the kind of education going on there. The teachers had the same kind of pride in what they were doing, as did the children. The individual child was the focal point; he was the learner and he wanted to learn.

The photographs, which are an integral part of this article, were done by Dr. Alfred Gross of A and I College, Kingsville, Texas during the teaching of the lesson.

Notes from a Math Specialist
Beryl S. Cochran
September, 1966

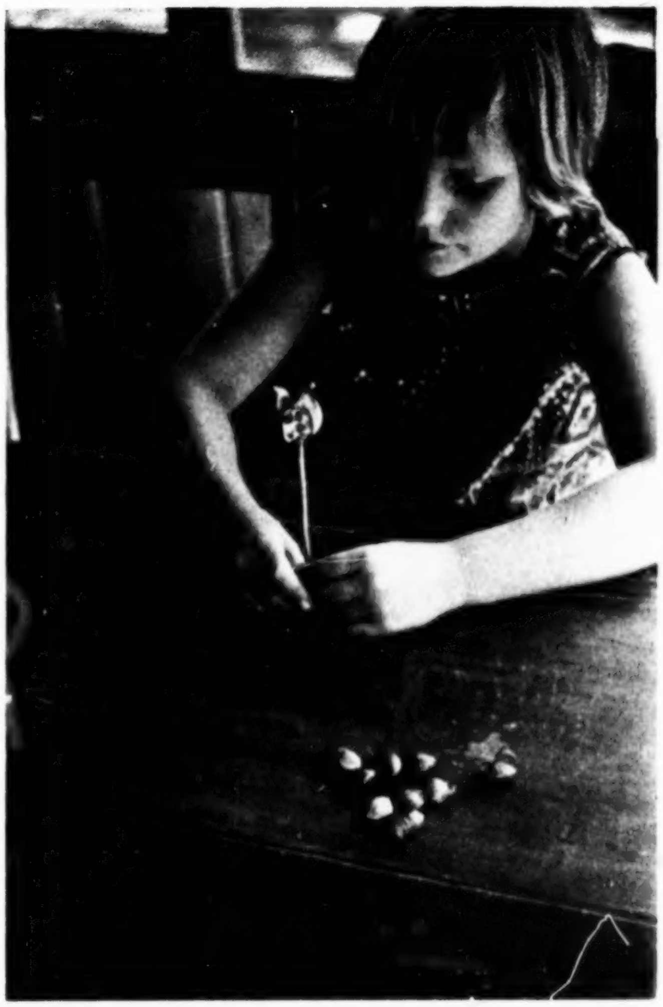
Exploring
FRACTIONS
with a six-year-old

I handed the little girl a seed pod that was divided into three sections. It was dried and opened up beautifully -- in three symetric sections.

Michele talked about the seed pod.

> "IT IS HARD OUTSIDE -- AND SOFT STUFF INSIDE. LOOK, THERE ARE LOTS OF LITTLE BLACK SEEDS. IT HAS THREE PARTS -- THEY ARE ALMOST EXACTLY ALIKE."

I took the cue:



"Michele, I call these sections and we say the seed pod is divided into thirds. Do you think you could divide this pile of plum seeds into thirds?"

"WHAT DOES THAT MEAN?"

"Count the sections in the seed pod."

"THREE"

"How many equal piles of seeds do you think you should have?"

"THREE EQUAL PILES.

O.K. I CAN TRY."

And she did.

She counted her equal piles

and had ONE

LEFT OVER.

"Here are some other things,"

and I pointed

to the Cuisenaire Rods.

"Can you find some thirds here?"

Ten minutes later -- after a great deal of trial and error thinking and doing --
she had a nice collection
of THIRDS.

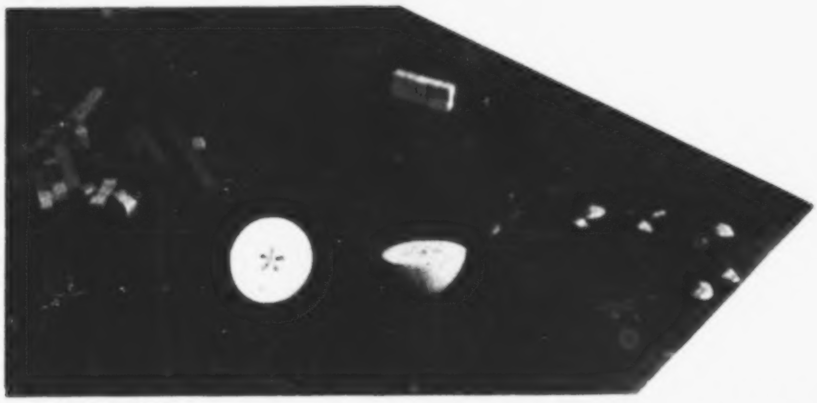
Michele had worked with the seed pod that was divided into three sections -- she had worked out thirds of ten plum seeds and thirds of various Cuisenaire Rods.

I told her that she was working with fractional parts of WHOLES. If it was ten plum seeds, we considered the ten seeds our WHOLE pile. If it was our seed pod we considered it the WHOLE pod.

Michele asked if she could work out some other 'fraction' things.

We found a pear

and cut it in half



We looked at the design

that the seed section made

Michele decided that

this must be called something like FIVERDS --

BUT we settled for the word FIFTHS.

She worked out many fifths with her

MATERIALS ...

as

she

ate

the

pear.

Notes from a Math Specialist
Beryl S. Cochran
November, 1964

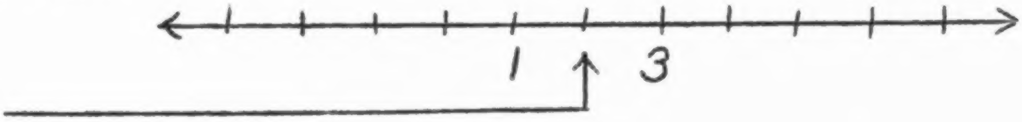
CHILDREN USE SIGNED NUMBERS *

The arithmetic of signed numbers is one of the "big ideas" in mathematics that I find young children almost compelled to invent for themselves.

While teaching in a first grade last month, I had drawn a line on the board.

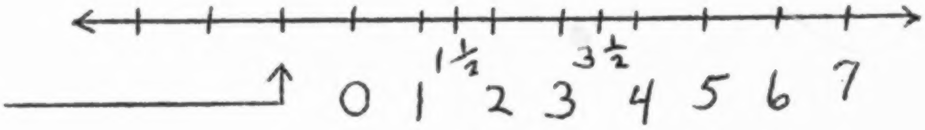
I had put some cross marks on it and labeled two points.

I asked, "What goes here?"



The children easily filled in the line, including zero and a few positive fractions. We even extended the number line to the right.

I soon asked, "What goes here?" I got the usual answer, "ONE."



I replied, "We have one here," pointing to the right of zero.

One child said, "I GUESS IT'S ZERO."

I replied, "You told me zero went here," pointing to the zero.

Since I don't mind waiting if it looks as if some thinking is going on, I waited two minutes or so, but nothing was forthcoming from the class. So I started to erase the board. With that, Roger, who had said nothing so far in this lesson, took his thumb out of his mouth and said in a youthful dialect that I could hardly understand: "OH, MRS. COCHRAN, LET'S FILL IN THIS PART." And he pointed to the line at the left of zero. "I THINK A BACKWARD ONE GOES HERE."

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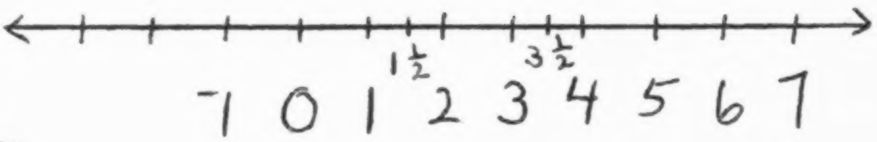
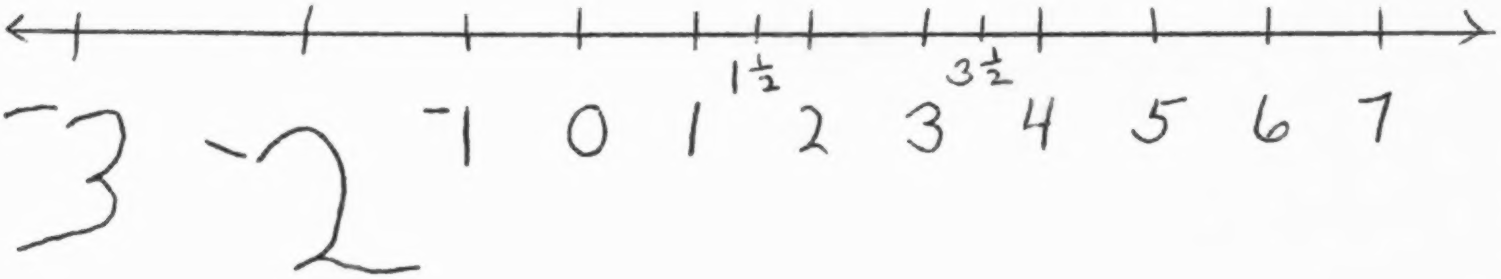
[*] This article appeared in The Arithmetic Teacher, Vol. 13, No. 7 (November, 1966), pp. 587-588.

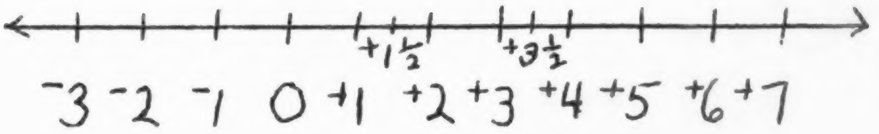
"I guess you are right, Roger. Mathematicians call it 'negative one,' and I'll write it this way." I put down the one and the high negative sign.



Roger then took the chalk out of my hand and proceeded to extend the number line to the left and filled in ⁻2 and ⁻3.

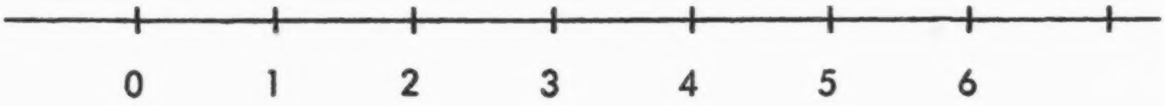


I allowed the irregular intervals that he had marked to stay on the board, but I did go on to explain that mathematicians call the numbers to the right of the zero position "positive," and denote this by a small plus sign up high to the left of the numeral. I then corrected my number line.



I felt that this was quite enough door opening (or rather, mind-opening) for one day, so we went on to do some arithmetic with our bags of washers.

How else could one have introduced the number line to a first grade? We could have opened our work books or filled in a work sheet where we would eventually find a number line that



starts with zero and ends with some positive integer and have the children fill in the missing numerals.

This would be a good lesson for practicing the handwriting of the numerals and to help train the child to follow directions and work neatly on a piece of paper. Some children might ask to extend the number line, and be allowed to; others might be encouraged to fill in a few fractions.

My interest seems to lie in trying to understand and cultivate another type of learning, a learning that seems to take place from within the learner -- a type of learning that Roger was ready for in this introductory lesson, and, as shown the next day, a type of learning several of the other students were also ready for.

The next day, the teacher of this class told me she was playing a game,

"WHAT HAVE I DONE TO YOUR NUMBER?"

She made up a rule, kept it a secret, and asked the children to tell her a number; she would use her rule and write the results.

	□	△	
They said, "THREE."	3	7	She wrote 7.
They said, "FIVE."	5	9	She wrote 9.
Joe said, "NEGATIVE TWO."			She said "WHAT"!
Joe said, "LET'S TRY NEGATIVE TWO."	$^-2$	$^+2$	She said, "All right, let me think. O.K., positive two."

The teacher was obviously an alert individual who was ready to let the children try a new idea.

Joe, who was not the child who had extended the number line the day before, had obviously learned from the experience, and now wanted to see if this mathematics stuff really checked out.

When I talked to him later, he told me that he figured his teacher's rule was □ + 4 = △ and if $^-2$, $^+2$ worked, "I COULD MAKE A LONGER CHART THAN ANYONE, BECAUSE I CAN USE ALL SORTS OF NEGATIVE NUMBERS, NOT JUST THE REGULAR OTHER ONES."

It is repeated experiences like this that make me feel signed numbers will be creeping into our teaching of arithmetic more and more as we learn to make use of the child's intuitive understanding of mathematics and its logical notations.

By allowing the student in his arithmetic work to use all the integers and visualize the number line to the left of zero as well as to the right, we are freeing him, rather than giving him more to learn.

I don't know if five other children in the class even cared if $^{-}2$, $^{+}2$ worked; but, as far as I'm concerned, it's worth taking the time to present such a lesson as the introductory number-line lesson just to see if some children won't push you, the teacher, into logical, yet unusual, mathematical situations.

What is it with a learner that after a period of time a concept has taken hold and grown? I might say "after a period of reflection"; but that would connote the idea of conscious thinking on the topic, and I do not believe this to be true with these young students. I rather feel that an electric spark has been sent out and some contacts have been made. There will be many a trial-and-error period to see which connections work; but once the spark has made contact and the individual learner is free to test his theories, real learning is taking place. Thus we can draw the children into the learning process instead of always being in a position of having to push them through it.

Notes of a Math Specialis
Beryl S. Cochran, 1965-6

PLACE VALUE IS DIFFICULT

Maybe it's impossible to teach, else why does it keep popping up in the guise of different bases at different grade levels?

Could it be that we are teaching it as a notation, not as a concept? Why should 101 look much different to a child of seven than 10 or 1001? This child has had very little experience with counting or even observing large numbers of things. I'm sure he is aware of the fact that there are more people in church or at the grocery store than there are around the supper table, or even the school room, but I tend to think that the young child does not come by his understanding of place value as naturally as he seems to come by the understanding of one-to-one correspondence; i.e., counting meaningfully.

1 stands for one thing ●

2 stands for one thing and one more ● ●

3 stands for one thing and one more and one more ● ● ●

etc.

but 100 just means lots, although it does have a name to these young children: it is called "One Hundred."

I have been known to have fourth grade children fill cloth draw-string bags with

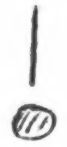
10 beans
100 beans
1,000 beans
10,000 beans

on the excuse that I want to take them into the first and second grades so the children can feel and hold these large numbers of counted things. However, I'm not really sure but that I do it to give these fourth-graders experience with these large numbers of things. It is in this fourth grade period that you see a lot of computational errors due to a lack of understanding of place value -- and to have physically worked with these counting objects does seem to help clarify this concept of place vlaue.

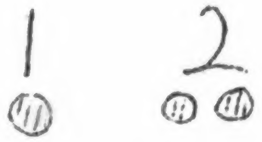
I have had experience with young children that makes me realize some children have an understanding of place value quite young, but I don't feel these understandings are universal.

One story that I like to tell happened in a living room at story time. A little five-year-old wanted a certain chapter read to her -- she was even able to look this chapter up in the table of contents. She discovered that her chapter started on page twenty-four. Then she hunted through the book for twenty-four. As she got to page thirty-one, she said, "Oh, my, I have to go back seven pages." Obviously this child had an understanding of place value and subtraction, and a reading ability -- and with no formal schooling to date.

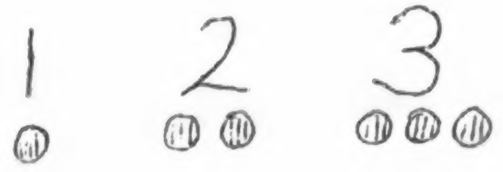
In a kindergarten class where I was teaching (in fact it was my first kindergarten lesson, and I was starting out carefully, tactfully, and easily), as the children sat on a rug in front of the blackboard, I'd ask someone to bring me one block, writing 1 and then drawing a large, filled-in circle under it.



Next I asked for two blocks.



Next I asked for three blocks.

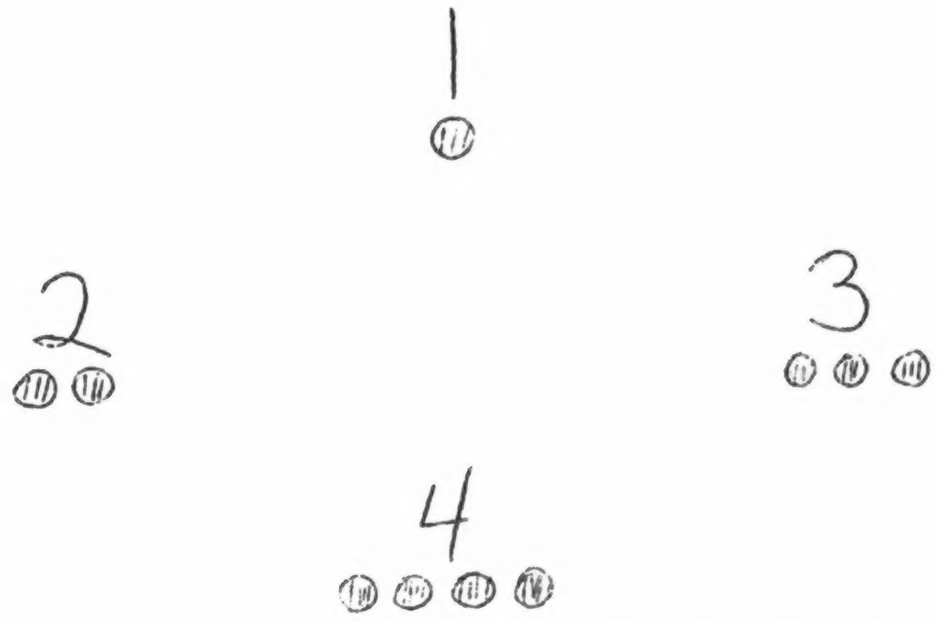


And I was about to ask some helpful child to bring me four, when I heard,

> "I don't know what she is doing. That's one hundred and twenty-three she has there."

Obviously this child had a real understanding of place value and not much understanding or interest in the dull sort of work I was doing.

NOTE: When I try this lesson now I'm very careful to write in odd spots all over the board.



I am sure, however, that these are the exceptions, and as we think in terms of teaching place value, we must remember we are using the same old ten symbols to stand for all sorts of piles or bags of counted things.

Teacher Training Notes K
Beryl S. Cochran 1965-6

A First Lesson in Place Value
or
"Bean Pasting"

Material: Tongue Depressors
Glue (Elmer's Glue in little bottles)
Beans

My standard bean-pasting lesson comes around Christmas of first grade; it is three or four months after the children have been in formal arithmetic classes.

They have been having many and varied experiences with the math lab material; they can make all sorts of shapes on the nail boards and reproduce them on dot paper. They can write the equation for a story problem: $3 + (2 + \square) = 8$. The students' handwriting is getting decidedly better, and his vocabulary is growing and becoming more precise.

It is time to take this difficult concept of place value by the horns. This has three presuppositions: (1) the children can count accurately to 20 or more; (2) they know the meaning of all the operations; (3) they feel a need to stop all this counting and grouping with loose things -- it is inaccurate.

The Lesson:

Teacher: I want to do something with the beans today. How many shall we take?

Joe: SEVEN. 7

Betty: THREE. 3

Heather: EIGHT. 8

Tim: NINE. 9

Patty: TWELVE. 12

Teacher: How many have we in all?

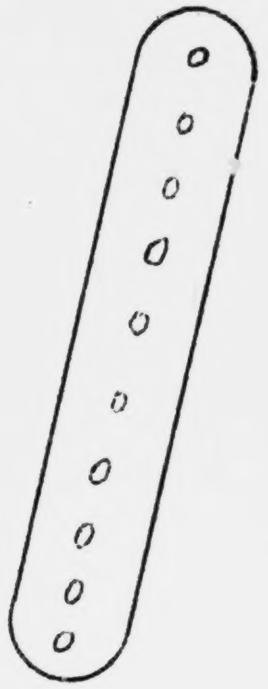
Class: TOO MANY.

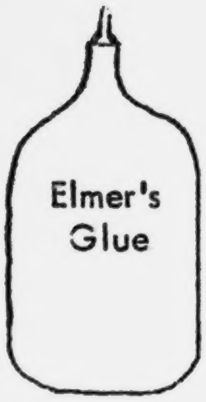
Teacher: I guess you are right. I have an idea. What does that "one" stand for in Patty's number? How many beans?

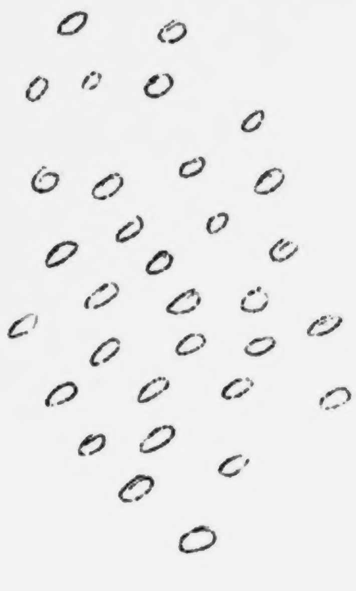
Glenn: TEN.

Teacher: Suppose we paste ten beans on these sticks and then they will always be there.

We pass out tongue depressors, several squeeze-bottles of Elmer's Glue, and a handful of beans. The children usually work in groups to share the glue. They drop ten drops of glue on the tongue depressor and then place a bean on each drop.







The making of the bean sticks is enough counting and confusion for one day. We leave them to dry and go to the library.

5 neg

Teacher Training Notes K-2
Beryl S. Cochran 1965-6

Bean Pods to Bean Sticks

This idea came to me while in Africa -- not the use of the tongue depressors, but the use of bean pods. I felt it was a brilliant idea and would certainly solve my first place value problems; however, with all the beautiful bean pods hanging from lovely trees, who could find one with ten beans? Not I. But the thought was not lost and it came to life in a first grade in the U.S.A. when the proverbial problem of place value arose again.

This bean stick idea has been very successful with first grade teachers; of course, they still bundle and group and make sets of things -- this is all necessary too. The concept is a difficult one and needs much reinforcement, but the bean sticks have an added dimension in that they can be perceived as either one ten, or ten ones.

I recently made a film for the Madison Project, "Operations of Arithmetic," where two first grade boys sitting side by side did have this diverse attitude toward the bean sticks. They were to take seven beans. Peter took a stick and held it with his thumb over three of the beans. His neighbor, Duncan, took a stick, counted down the row of beans until he reached seven, saw there were more and put it back (remember the children had made these sticks). He then took seven loose beans. When will the Duncans react like the Peters? Who knows, and why the rush? It is Duncan who wants a friend to do bean arithmetic with him all day long.

Second Lesson with Bean Sticks

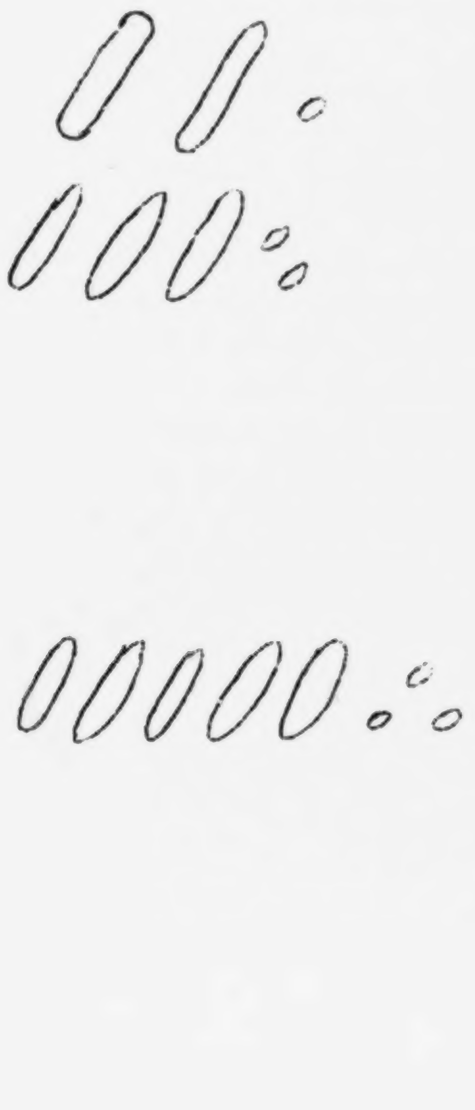
The class has tables in groups of three; we put a box on each table with loose beans as well as bean sticks in the center. The child is free to choose. He can pull out either loose beans or sticks. If he is having trouble, we will direct him and help him group, but I'd rather he would use his own instincts first.

Many first-graders like to use the box of beans and bean sticks and a third grade book and do some of the problems they find there.

We find the children soon making notes when they want to add large numbers.

21
+ 32

53

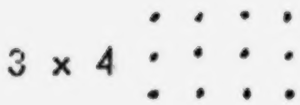


Teacher Training Notes, K–2
Beryl S. Cochran, 1965–66

MULTIPLICATION
Math Material Center

WASHERS

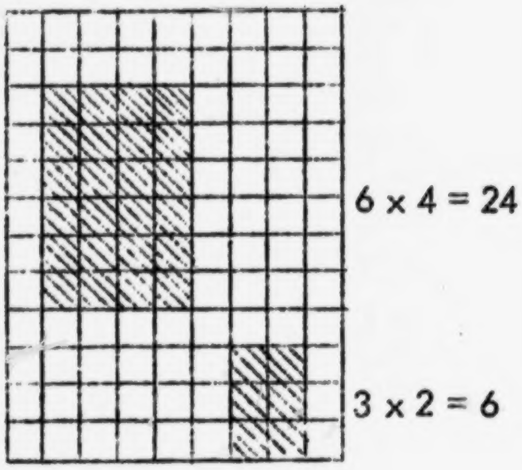
The introduction of washers into the classroom allows the children to manipulate materials into arrays and thus picture their multiplication problem:



The first grade children enjoy a lesson on multiplication every once in a while. If it is introduced in first grade we often use a red chalk to make the times sign. Young children are apt to read it as a plus if we are not very careful.

SQUARED–OFF BOARD
and
GRAPH PAPER

The squared–off–board and graph paper allows the elementary teacher to carry the operation of multiplication ahead freely and comfortably. The students can really do most of this work on their own once the teacher has played the game "what goes here" several times. She might do it with the full class, or with small groups of children who seem to need a bit longer to discover the pattern. Once the child has a means of attacking the problem and is able to picture multiplication as a colored–in area, he is free to develop his own multiplication chart.



	1	2	3	4	5	6	7
1							
2							
3		6					
4							
5							
6				24			
7							

What Goes Here

We draw a chart, fill in a few numbers so that students can see what operation we are working with, and we divide the class in half. The teacher then points to an empty space and says, "What goes here?"

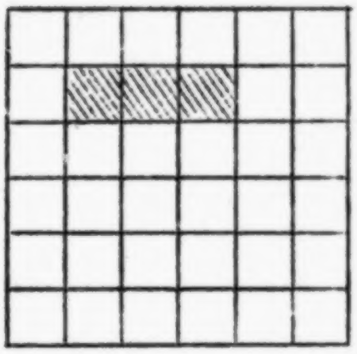
Notes of a Math Specialist
Beryl S. Cochran, 1965-66

EXPLORATION WITH MULTIPLICATION
in Second and Fourth Grades

Having been asked when in the school curriculum one should introduce multiplication to children, I suggested third grade. Later I visited a school where I was scheduled to teach a second grade class.

> Why not give multiplication a try with this group of young children -- certainly they can all count and do some of the operations.

In fact, I reasoned, it should be feasible to teach multiplication along with addition and subtraction. Certainly if children can count, we can give them an approach to multiplication that will have meaning to them. I decided to give it a try with this second grade class.



At the chalkboard I drew a grid. I colored in three squares horizontally. I asked the children to describe what I had done. Needless to say, they thought this was a very silly question and giggled as they answered,

"YOU COLORED IN THREE SQUARES."

"I will write the mathematical sentence for that." I wrote the one first, then the times sign, being careful to make it very dark; then I asked, "How many squares did I color in?"

1 X

"THREE," the class answered.

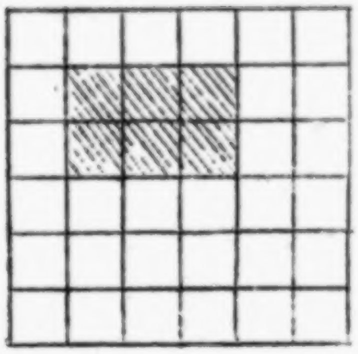
I then wrote the three and the equal sign, and asked, "How many squares have I colored in, in all?"

1 X 3 =

"THREE," the children told me.

1 X 3 = 3

I colored in three more squares directly under the first set and asked the question again, "How many squares did I color in this time?"



Several students spoke up to tell me I'd colored in six in all. "And how do you suppose I'd write that?"

Many of the children wrote six on their desks or in the air to show me. I had meant to ask, "How shall I write the complete mathematical sentence, so I said,

"Yes, that's six -- and this is a picture showing two times three equals six."

$2 \times 3 = 6$

I colored in three more squares and asked, "How many times have I colored in three squares?"

"THREE TIMES."

3×3

"I'll write the sentence. What does it equal?"

"NINE."

$3 \times 3 = 9$

I was always very careful to mark a strong X for the times sign. I think children are apt to get this sign confused with the plus sign when it is first used. I write it very distinctly and slowly.

We carried this multiplication ahead slowly and easily until I felt the students were comfortable with the mechanics of this approach. Then we did several more examples exactly this way. My question was,

"How many do you want me to color in now?"

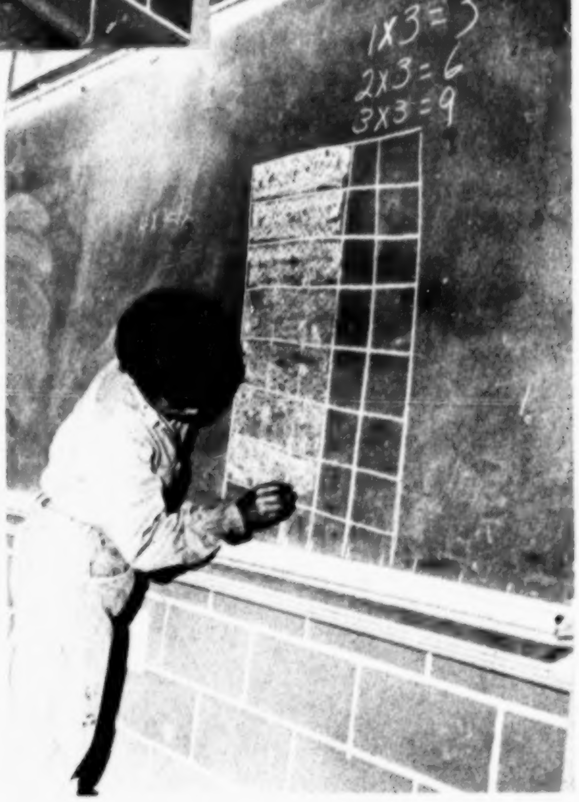
Someone said, "SIX."

So I colored in six; then right under it I colored in six more and asked, "How many times do you want me to do it?"

Four Times Three
is
Twelve

1x3=3
2x3=6
3x3=9

Six Times Three
is
Eighteen


1x3=3
2x3=6
3x3=9

SO - --
Seven Times Three
must be

Nineteen

Twenty

TWENTY-ONE

I accepted the number four and colored in the area. Then I asked, "How many did I color in the first time?"

"SIX."

$1 \times 6 = 6$

"How many did I color in the third time?" As I spoke, I wrote

$3 \times$

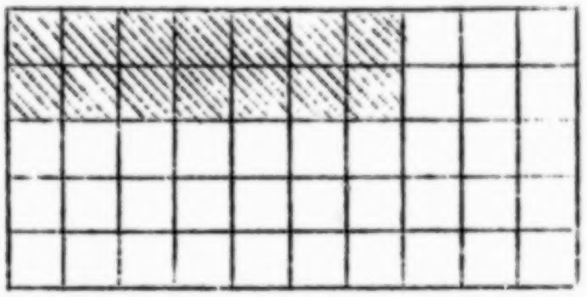
As I wrote, the children called out: "YOU COLORED IN SIX MORE, BUT THERE ARE EIGHTEEN IN ALL." So I wrote

$3 \times 6 = 18$

I asked if someone wanted to write the mathematical sentence for the whole picture and several children volunteered.

We did several other examples and at one time had some questions on the difference between two times seven and seven times two.

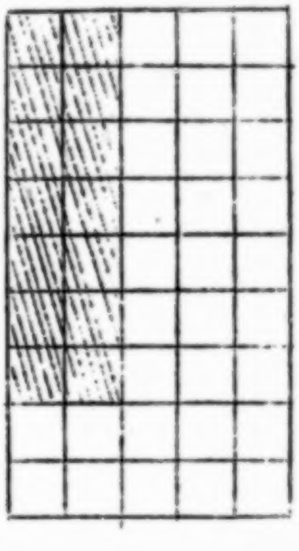
One child, Ricardo, showed the class a picture for two times seven.



$2 \times 7 = 14$

Another child, Mary, showed them the picture for seven times two.

$7 \times 2 = 14$



Many of the children in this second grade class went on during the next two weeks to make their own multiplication books. That is to say, half-inch graph paper was available to them in their classroom. During their free time they filled in many series of squares and wrote the multiplication example that went with it. They kept a large squared-off board which the principal had made. It was of common masonite, painted over with blackboard paint, with white grid lines marking off three-inch squares. This board was kept at the front of the room where the children then filled in what we would commonly call the Multiplication Table. They did not do this in a systematic way (one times two, one times three, one times four), rather, they filled in any of the combinations that they worked out for themselves, even up to 15 x 7. Within three weeks or so the children themselves had begun to learn certain "facts," and they were well on their way to learning the multpilication table.

This seems to be a very easy introduction to multiplication. Professor Andrew Gleason has said that probably area should be taught before multiplication, but I rather feel that they should be taught together. As I left this class, I went into a fourth grade class and was feeling them out, both on their multiplication table and their approach to multiplication, and then to area. I was able to do a great deal of work with this fourth grade class from almost the same approach. I found the children did not have a picture of multiplication in their minds -- they seemed to be relying almost entirely on memorized facts. I gave them the example 15 x 21; half the class was able to do it by one method or another, but they were very unsure of themselves.

We then worked on 105 x 201. I picked these two large numbers because we had just completed the example 15 x 21, and since zero and place value seems to be one of the real bug-a-boos, I thought this would help point up any notational or computational misconceptions.

The teacher made both plain paper and one-fourth inch graph paper available to the group and assured them that they had ample time in which to work on this problem: "So let's just see what you can do." Needless to say, the attitude of the teacher is what made this a successful lesson, for as we both walked around making suggestions and answering questions, we were able to diagnose many erroneous ideas about place value and the operation of multiplication. Children started working together. I find this often the case and very advantageous, for a conversation with an interested, involved partner helps clarify one's thoughts.

It was a busy morning and part of the lesson was very noisy. It was probably at this time that most of the learning was going on. By the time I left this classroom, the children had done many a complicated multiplication problem; one of the groups of children had decided to change their scale for the hundred and tens and unit part of the problem.

Giving the children experience with area at this time seemed to be very beneficial. They could now picture multiplication, and it seemed to facilitate their accuracy with the standard algorism that their text book used, which they went on to develop by themselves over the next month. This lesson also seemed to help them move ahead by leaps and bounds in the text.

Since that day, I have given a lot of thought to multiplication. This fourth grade class seemed to me to be the obvious result of what I call "multiplication table philosophy." As anyone in modern math knows, one of the main questions that comes up at any meeting or informal get-together is, "What about the multiplication tables?" I, for one, believe that children should in time memorize these facts, but I want the students to understand what multiplication is first. I want them to have a way of attacking the example. There are only four concepts involved in such an example as:

105 x 201

or even 1005 x 2001

or even 10005 x 20001

These are exactly the same concepts. The first two are counting and numeral recognition; many children come to first grade having mastered these for two and five. However, the next two, PLACE VALUE and the OPERATION MULTIPLICATION are more complex, and as with any concept, these must be developed slowly, visually, and with involvements coming from the individual learner.

The teacher should not look for mastery of any concept until the child has been given many free activity experiences. The teacher will then be in a position to observe the individual child as a learner and thus help him as an individual thinker.

Once the individual learner is comfortable with these concepts, he is free to attack any multiplication problem and develop his own algorisms, which naturally enough usually turn out to be the same one presented in the various texts currently on the market.

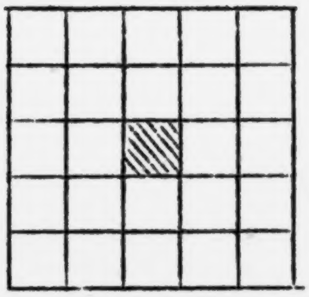
With some luck, however, you might get one or two new algorisms, which always lend a spark of excitement and a feeling of discovery for the class as well as the teacher.

Teacher Training Notes K-2
Beryl S. Cochran 1965-6

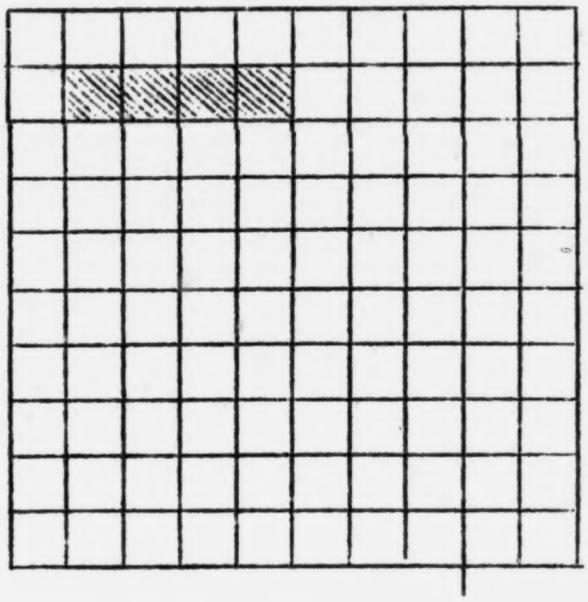
MULTIPLICATION
An Introductory Lesson

Material: Squared Board
Graph Paper

Teacher: If I call this square one unit,



how many units did I color in?

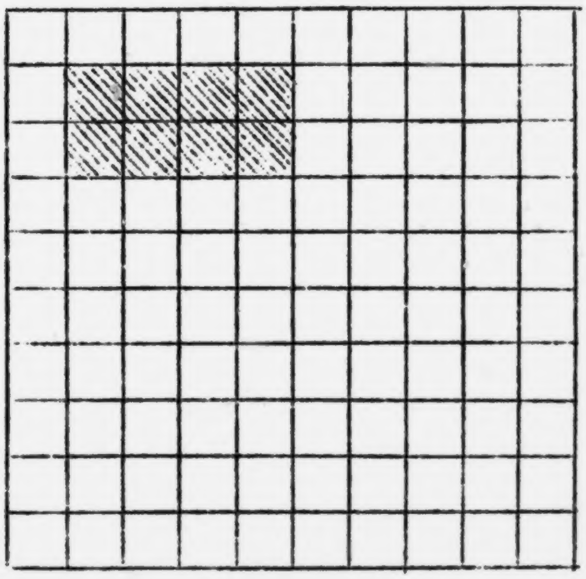


Class: FOUR.

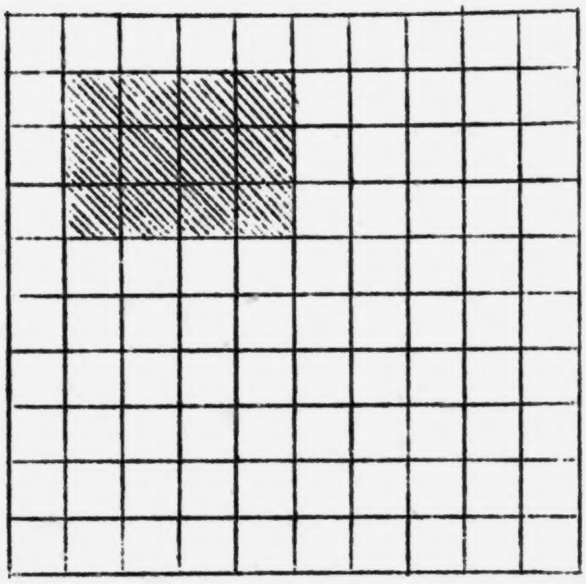
Teacher: How many times?

Class: ONE TIME.

Teacher: I'll color four units another time.



Teacher: I'll color four units in again? How many times have I colored in four units?



Class: THREE TIMES.

Teacher: I'll write the multiplication story this way.

$$3 \times 4 =$$

I make a very dark times sign, especially heavy, so the children won't mix it up with a plus (+) sign.

Teacher: What is three times four?

Class: TWELVE.

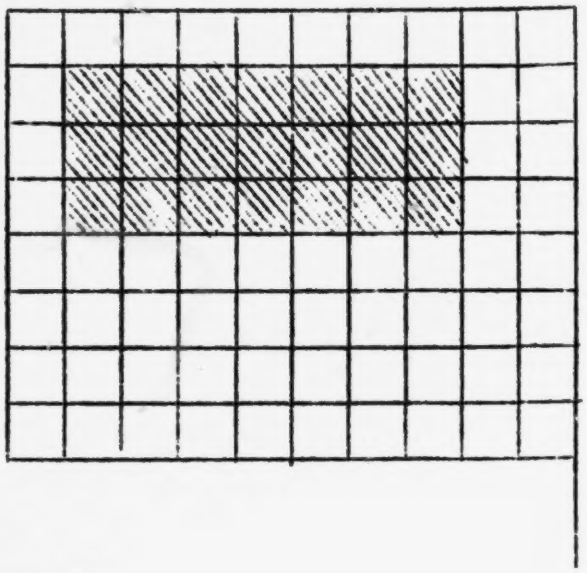
$$3 \times 4 = 12$$

The class is able to see the general pattern of the work, now, so I try to involve them more, yet continue the lesson in the direction I want to go.

Teacher: How many shall I color in now?

Class: SEVEN.

Teacher: All right, I'll do that three times.



Teacher: How many times did I color seven squares in?

Class: THREE.

Teacher: So I'll write the equation this way.

$3 \times 7 =$

Teacher: How many squares are colored in, in all? Who wants to come up and count them?

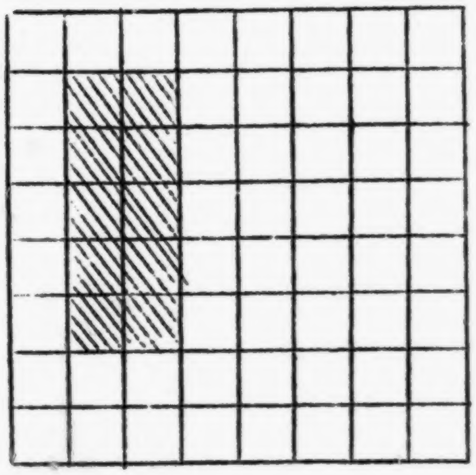
What is three times seven?

Class: TWENTY-ONE.

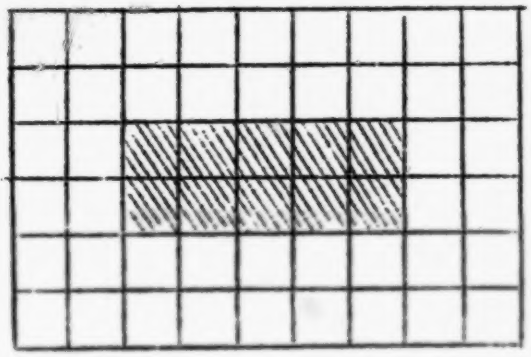
Teacher: Who wants to write the multiplication equation for this picture?

John says it's $5 \times 2 = 10$.

Betty says it's $2 \times 5 = 10$.



The picture for Betty's equation would look like this.



Many children in second grade will point out, and you can often lead them to verbalize

$$2 \times 7 = 7 \times 2$$

$$\square \times \triangle = \triangle \times \square$$

which has the formal mathematical term of Commutative Law for Multiplication, but if you are doing any work with identities, you might prefer to call it by the child's name who discovered it.

Susie's Law

$\square + 1 = 1 + \square$

By now the children have seen several examples and would like to do some work on their own. I hand out squared-off paper and write an equation for the first example. The children complete the equation and make the picture on graph paper, and as I walk around the room I can tell who needs some help. And those who have already grasped the concept are free to make up their own examples and draw their own diagrams.

Betty's Law

$\square + \triangle + 7 = \triangle + 7 + \square$

At this time we often start a multiplication chart that the children can add to as they make up their own examples.

	1	2	3	4	5	6	7	8	
1							7		
2		4							
3				12					
4						24			

NOTES FOR A CHAPTER

ON

COUNTING AND OPERATIONS

CHILDREN LEARN TO COUNT

We are often confronted with the query:

"What do you do with children who can't count?"

First grade teachers tell me that the answer to this question is, "We teach them to count."

The children are matching the spots on the die to a card with the same number of spots.

Many varied and informal experiences directed by the teacher and other students will help the non-counters learn to count.

The secret to this learning seems to be games and materials with which to work freely, giving the child free experiences before rigorous teaching.

The old game of hopscotch.

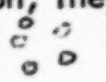
Teacher Training Notes, Level I
Beryl S. Cochran, 1963-64

COUNTING AND OPERATIONS

COUNTING

Bag of
Counting
Things

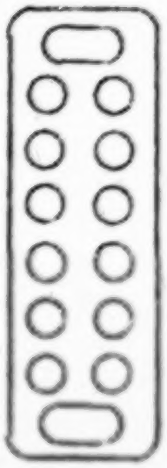
Counting is one of the earliest experiences that a child has with mathematics. Oneness, twoness, and threeness are concepts that are important, and these concepts come only after many and varied experiences with counting and grouping things.

It is the belief of the writer that many children come to school able to count, and with an understanding of one-to-one correspondence. They don't just utter words and sounds. If children are given MATERIAL THINGS to count during an arithmetic lesson, they will group and learn to eye count; i.e., count at a glance that  is 5.

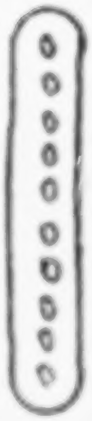
OPERATIONS

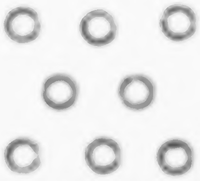
While it has been traditional that addition be the first operation children encounter formally in arithmetic, I have often introduced division first, and then subtraction, since these are the natural situations for the child to have encountered. He has often shared (division) with others, and certainly many things have been taken away (subtraction) from him.

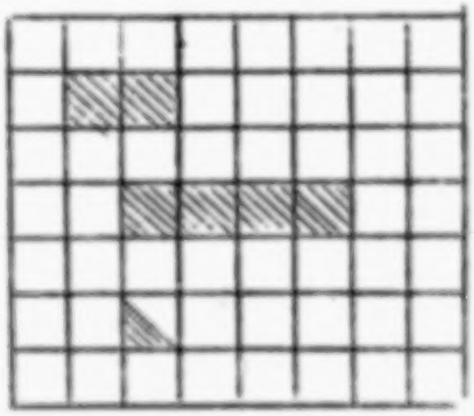
Our lessons always build on experience -- whether we give this experience to the children, or they obtain it in a more informal way.

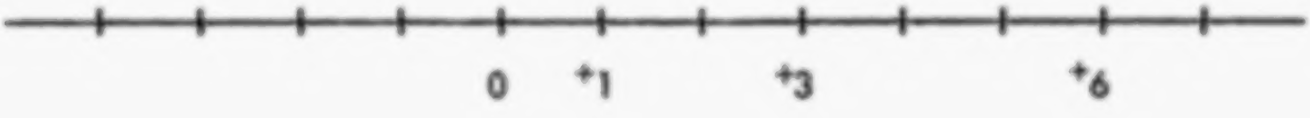

Kalah Game

Whether division, subtraction, or addition is first introduced to the child in class, there seems to be no difficulty in his ability to do all operations within a day or so of each other, when given counters with which to work. I feel very strongly on this point and urge all teachers to try the approach of using concrete objects.


Beans and Bean Sticks


Washers or Bottle Tops


One-half inch squared paper



Teacher Training Notes K-2
Arithmetic: Second Grade
Beryl S. Cochran, 1966-67

The Reading and Writing of Mathematical Statements

Materials: Counters
Kalah Boards
Small Index Cards
Small arithmetic paper
Pencils
Baskets to hold extra materials for each group of six

Much of the work in the second grade year will be concerned with giving the children experience in

1) the reading of numerals

2) the reading of the four (+, −, x, ÷) operation signs; the children will read the mathematical sentence and work out the problems with concrete objects

3) brackets and parentheses will be used to show what groupings are intended

4) the children will be encouraged to write and work out their own examples.

In general the work will consist of reading the arithmetic statement much as they would an English sentence. The children have had many an hour of working with counting things in their first grade work -- many a formal and informal experience -- and now during this second school year we hope to carry them much further with this arithmetic work.

Arithmetic only tends to become difficult for a child when it is put in the abstract stage -- on the printed page -- before his learning process has carried him there.

We must approach teaching from the learning point of view. We want to move the child from the concrete -- to the symbolic -- to the abstract.

Teacher Training Notes
Beryl S. Cochran, 1966-67
Second Grade

LET'S READ ARITHMETIC SENTENCES

Materials: Counters

Recognition, action, and ability to work out simple arithmetic examples involving the numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, and the operational signs +, ÷ , −, X.

Since the children are doing a great deal of their arithmetic work with concrete objects, it is easy to approach many of the math sessions with the same philosophy as one approaches a reading lesson. A statement is written, the symbols are recognized and read, a thought is conveyed. Sometimes the complete thought is grasped, such as $4 + 1 = 5$, but a statement that is a bit more complicated might have to be worked out -- but thinking time is learning time. So give out the counting objects, put the following equations on the board, and give the children time to think and work things out.

$(3 + 7) + 2 = 12$

$(2 \times 4) = 8$

$(7 - 2) = 5$

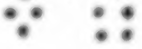
$12 \div 4 = 3$

$(2 \times 3) - 2 = 4$

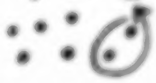
$17 \div 3 = 5$ and two left over

I like to think of the operations in the following ways:

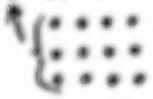
Addition: Collecting together $3 + 4 = 7$



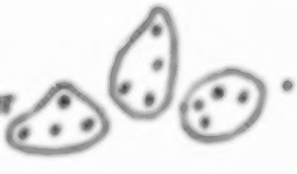
Subtraction: Taking away, losing $7 - 2 = 5$



Multiplication: Arrays $3 \times 4 = 12$



Division: Sharing $13 \div 4 = 3$ and one left over



"Do you want a hard one or an easy one?"

$$3+(2\times 7)=\underline{\qquad}$$

$$9+3=$$

You can hand out a whole variety of problems to the children, depending upon *their ability* and what they seem to be ready to attack today.

$$\left[\left(\frac{1}{2}\times 12\right)+6\right]-2=$$

Individual Work
Washer Arithmetic

Third Day

This was done in a special class of fourth grade children.

Washers and beans were both available.

$$\frac{3}{1}=3$$

$$\frac{4}{\frac{1}{2}}=8$$

$$3+(2\times4)=8$$

$$4\times3=12$$

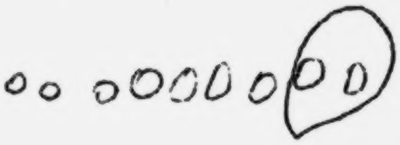
$$\frac{10}{2}=5$$

$$\frac{24}{6}=4$$

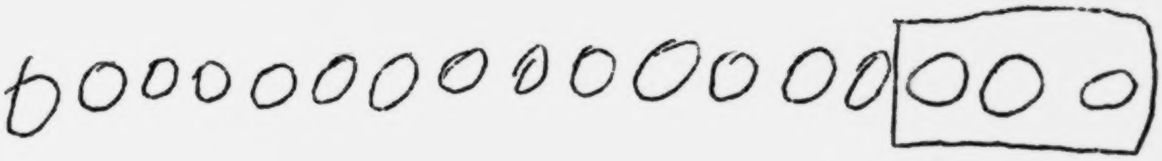
$$\left(\frac{1}{3}\times12\right)\times3=12$$

Melvin Smith

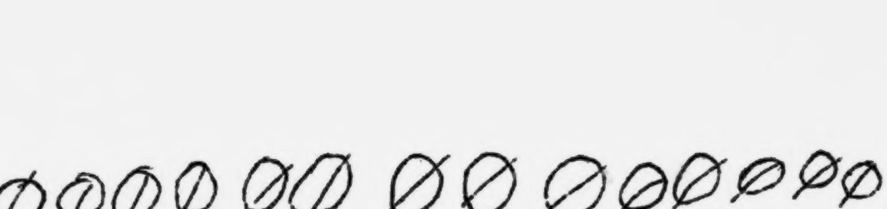
9-2=7 9-2 =7



17-3=14



17-3=14



21-18=3

Work out without Beans

$20 \div 3 =$

$\frac{20}{3} =$ 6 left over
2

$\frac{21}{3} = 7$

$$\frac{12}{4}=3$$

$$\frac{12}{2}=6$$

$$\frac{12}{6}=2$$

Tomseclow

$\frac{16}{10} = 1$

Left over
6

$\frac{27}{12} = 2$

Left over
3

$\frac{72}{5} = 14$

Let Tover
2

$\frac{29}{13} = 2$

Left over
3

$\frac{18}{9} = 2$

Left over
0

Tom Seclow is a second grader.

He seemed to like to work out hard problems with the counters -- then write the problem down.

He seldom drew diagrams of dots.

Jimmy

$\frac{10}{\frac{1}{2}} = 20$

$\frac{17}{5} = 3$ 2 left over

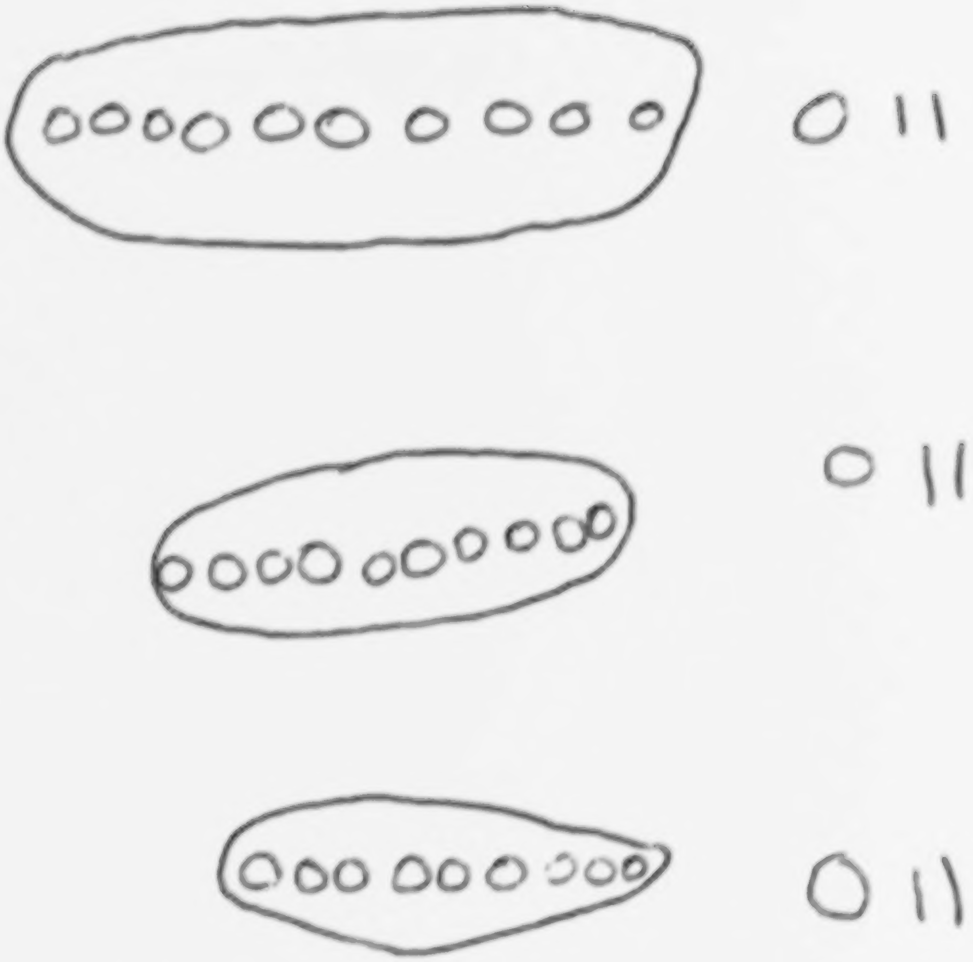
$\frac{18}{6} = 3$

$\frac{15}{5} = 3$

Three Times Eleven

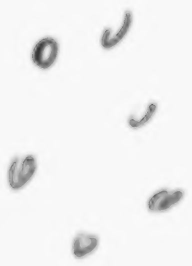
3 X 11 =

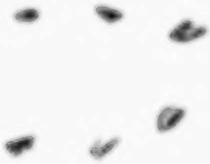
Gamalier

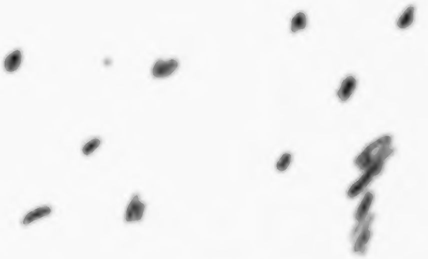


11 x 3 = 33

$6+(2 \times 3)+(10-1)=21$







M a r K

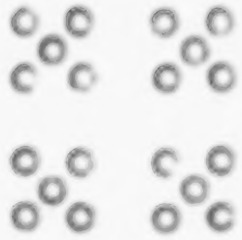
$8+2+1+9+1=21$

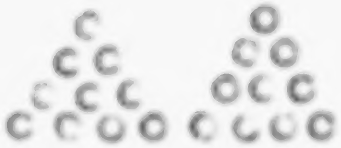
Second Grade
Beryl S. Cochran, 1966-6

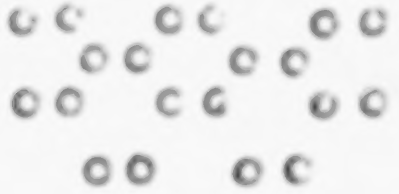
THE USE OF COUNTERS IN MATH
Second Grade

This lesson began by giving every child a bag of counters. These are common washers found in any hardware store. Each bag contains 20 to 30 washers -- the children enjoy using these washers as counters which help them in their mathematical computation.

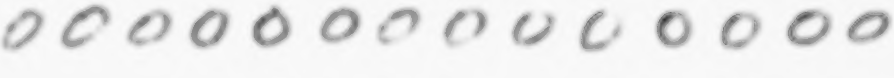
The teacher asked the children to group 20 washers so that she could quickly count them to see if each child had the right amount. As she walked around the room, these are some of the groupings that appeared:



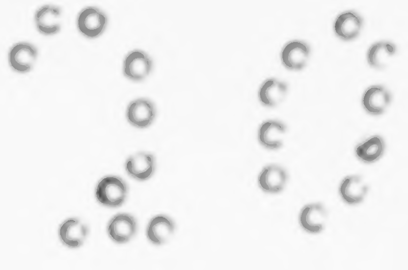




The teacher then drew some washers on the board and pointed out that these can't be counted as quickly, since they are not in groups.



One child grouped his this way and said, " YOU CAN REALLY TELL HOW MANY WASHERS ARE HERE!"



Since the children were very comfortable using the washers, the teacher moved on to some easy addition and subtraction equations:

$8 + 2 =$

$9 - 3 =$

$8 + 7 =$

$17 - 9 =$

$3 + \underline{\quad\quad} = 8$

$\underline{\quad\quad} + 7 = 15$

$10 - \underline{\quad\quad} = 3$

$18 - \underline{\quad\quad} = 10$

They needed very little help or explanation to solve these equations, so we then tried some complex addition and subtraction, using parentheses. The children have no trouble when using the parentheses, as they seem to naturally say to them, "Do me together."

$(3 + 4) + 5 =$

$(17 - 8) + 2 =$

$(4 + 3) + \underline{\quad\quad} = 15$

$(12 - 3) - \underline{\quad\quad} = 14$

$3 + (2 + 7) = \underline{\quad\quad}$

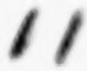
$\underline{\quad\quad} + (5 + 7) = 14$

The next part of the lesson was on equations where the children did not have enough counters or washers. The teacher put this equation on the board and asked the children to find the answer: $(4 + 9) + 9 =$

As she walked around the room it was interesting to see how the children reacted to this and what they used to solve the equation.

Some of the children used fingers

pennies

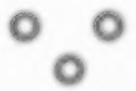
crayons 

cut up pieces of paper

The children who at first just sat and did nothing when they had used all their counters, were prompted to think when they saw how the other children had solved their equations.

The teacher then introduced the multiplication sign and had the children take 3 counters.

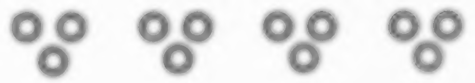
X



Then she wrote the numeral 3 on the board.

3

She asked the children to take three counters four times.



"This would be written like this and it says "FOUR TIMES THREE EQUALS."

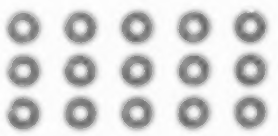
4 X 3 =

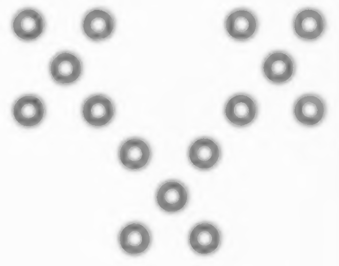
Another way to help the children understand the "times" sign is to read the equation as "four groups of three."

The teacher put a new equation on the board.

3 X 5 =

"Let's see if you can solve this equation." As the children were busy making their groupings, the teacher walked around the room to observe their arrays or grouping patterns. There will probably be two different grouping patterns or arrays.





"What is the answer to that equation?"

The children say, "FIFTEEN."

The teacher put on the board the two different arrays she had observed and said, "We have two different stories here, don't we?

One says 3 X 5 =
or 3 groups of 5

and the other says 5 X 3 =
or 5 groups of 3."

"Which of these stories is like ours?"

The children answered, "THREE TIMES FIVE EQUALS."

"What would the other grouping or array tell us?"

The children answered, "FIVE TIMES THREE EQUALS."

"Do we get the same answer for both?"

The children answered, "YES."

The class went on to other multiplication equations and did them in this way, with many arrays for each equation. We took two arrays of the same story and discussed the two "stories" the arrays told us, as on the next group of equations:

$(2 \times 6) + \_\_\_\_ = 16$

$(3 \times 5) + 5 =$

$(4 + 1) \times 3 =$

There are two ways of approaching division, as with multiplication. The children all took 12 washers, and while they were busy doing this, the teacher walked around and checked quickly to see if they all had 12.

She wrote this equation on the board

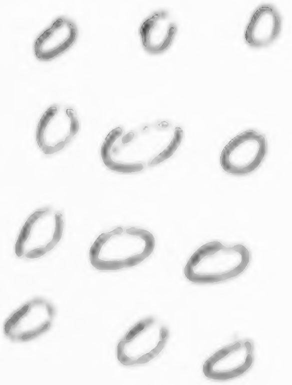
$12 \div 4 =$

"Now divide your washers into 4 equal groups."

The children grouped their washers. Some had a little trouble and looked to their neighbor for help.

"How many do you have in each group?"

The teacher called on one of the children that she had observed dividing her washers correctly to come up and draw on the board what she has done.



Another way to introduce division is to have the children take 12 washers or counters and then ask them to divide their washers into groups of four. Some of the children will have trouble again and look to their neighbor for help.

$12 \div 4 =$

Once the children were comfortable doing the division, the teacher gave them some equations that introduced other mathematical operations.

$(4+7) \div 3 =$

$(3+4) - 4 =$

The next part of the lesson had equations that would have left-over amounts or remainders. The teacher put this equation on the board:

$14 \div 3 =$

The children were not told that there would be a left-over amount or remainder, only that they should make equal groups to find the answer.

The children said, "WHAT DO WE DO WITH THE 2 THAT WON'T FIT?"

The teacher said, "Do you think that the 2 is part of the answer?"

The children answered, "YES."

The teacher said, "How many washers are in each group?"

Children: "FOUR."

Teacher: "All right, let's write it just that way."

$14 \div 3 = 4$
and 2 left over

The children did several more division problems this way, and when writing the problem on the board, the teacher used both division signs interchangeably.

$12 \div 4 =$

$\frac{12}{4} =$

NOTES FOR A CHAPTER

ON

MEASUREMENT

CHILDREN LEARN TO MEASURE

How wide is the table?

How tall is a first grader?

How many
little cubes
in a big cube?

What UNIT shall we
use to measure this
train of blocks?

Teacher Training Notes K-2
Beryl S. Cochran 1965-6

MEASUREMENT

Measurements are made to answer such questions as:

How tall is a first grader?

How tall is the flag pole?

How long does it take to eat lunch?

How heavy is this rock?

How many square tiles in the floor?

To measure we need a suitable UNIT.

Length is measured in LINEAR UNITS.

We use something that measures in a straight line -- a straw, a centimeter rod, a piece of string, a ruler, a tape measure.

Area is measured in SQUARE UNITS.

We use geo-boards, graph paper, square pieces of paper, floor tiles, a face of a cube.

Volume is measured in CUBIC UNITS.

We use kindergarten blocks, cubes, centimeter rod cubes, hollow plastic cubes that may be filled with small wooden cubes or water, baskets, containers of all sorts from home.

Weight could be measured in pounds, tons, ounces. Time could be measured in hours, seconds, years, decades. Temperature could be measured in degrees.

The important thing in measurement is to decide on an appropriate UNIT for your project. Measurement is always expressed as a multiple of some unit and the answer is always so many units plus some fractional part of the unit.

A first lesson in measurement is almost a gold mine for initial experience with concepts that will naturally become more and more refined as the child moves through the grades.

The main concepts we would like the teacher to be aware of are:

UNIT

ESTIMATION

ACCURACY -- leading to precision

The second grade teacher will be a bit more demanding of accuracy; but remember, we have many a year to lead the child to a proper concept of precision. Even now as I watch on television our progress in space, I'm awed by the precision required and OBTAINED by our space program. My concept of precision is still being refined. The following article which appeared in the St. Louis Post-Dispatch (Monday, March 6, 1967), p. 12 A., is a good example of the accuracy being obtained.

ERROR IN DISTANCE TO MOON IS CUT TO LESS THAN 150 FEET

© 1967, New York Times News Service

WASHINGTON, March 6 --

The question posed by the jazz tune "How High the Moon?" has been answered to an accuracy of less than 150 feet.

Space scientists announced the new accuracy figure yesterday in Washington and in California.

Because the moon lies about 240,000 miles from the earth, the new figure represents an accuracy of about one part in 12,000,000.

Until now the uncertainty has been about 6500 feet -- a little more than a mile. That error is enough to account for three crashes of the Soviet Union's Luna spacecraft in 1965 if they did not carry radar to control the final moment of descent.

Russia has not said whether these Lunas carried altitude-measuring radar.

In a year or so the error in distance may be narrowed to 15 to 30 feet, partly through radio tracking of the U.S. Lunar, Orbiter spacecraft. This would give Apollo astronauts more accurate flight plans for their 65-hour journey to the moon, requiring smaller steering corrections en route.

The latest refinements were made by J. Derral Mulholland, an astronomer, and William L. Sjogren, a senior research engineer, at the California Institute of Technology's Jet Propulsion Laboratory in Pasadena.

Notes of a Math Specialist
Beryl S. Cochran
November, 1965

MEASUREMENT

A Suggested First Lesson in Linear Measurement

Children at this age seem to like "personalized things," so we decided to measure their own height in some common linear UNIT: for this particular lesson we decided to use the common milk straw which was obtained from the school kitchen.

> Teacher: This straw will be my UNIT -- let's measure how tall Betty is. Betty, if you'll stand up here I'll measure up two of these straw units on your leg, and then we can estimate how tall you are -- before we actually measure you.

Betty came to the front of the class, and we measured from the bottom of her shoe -- one straw unit, two straw units. Since I also wanted the class to have some experience with the word ESTIMATE, I said:

> Teacher: Let's estimate. John, will you guess how many straws it will take to make a line of straws as tall as Betty?
>
> John: SIX.
>
> Teacher: I'll write the estimations here. How many do you think, Pat?
>
> Pat: FOUR.
>
> Joe: I THINK FIVE.

ESTIMATIONS
6
4
5

> Teacher: Pat and Bob, will you come up and measure Betty on the door jamb? Pat, you hold the straw, and will you, Bob, put your finger carefully at the top of it so Pat will know just where to start the second unit. Let's count while they measure.
>
> Class: ONE, TWO, THREE, FOUR, FIVE, SIX -- NO, IT'S NOT REALLY SIX, BUT IT'S MORE THAN FIVE.
>
> Teacher: We'll call it five and some more.

This is always a decision point in the lesson, for some child will always suggest five and one-half, no matter what the fractional part. With first graders I usually ignore it and keep the lesson focussed on measuring with a unit; but with second graders who should have had much more experience with measurement, I take this cue from the class, for here the fractional part becomes very important to the child who wants full credit for his height. One-half of a straw is certainly not the same as one-fourth of a straw, and here is a natural lead to fractional work.

The First Grade Lesson continues:

> Teacher: Let's put some paper around on the wall so we can mark your height, and you can measure your own height. But make an ESTIMATE first and write it to one side. Then choose a friend to help you measure accurately.

Now comes noise and confusion, but I hope learning also. It is certainly personalized and individualized. Paper is put up, or, if we are lucky enough to have lots of blackboard space, we USE IT -- it works beautifully. Each child is backed up to the wall, and we mark with a magic marker or chalk where the top of his head comes. He then gets his straw, chooses a friend, estimates his height, writes it beside his name and his height mark, and then proceeds to measure as accurately as he and his friend can manage.

A student in Virginia gave me a hint which helped this measurement lesson. He pulled out of his desk a well-used, well-hidden piece of clay and used it to mark the top of each straw length. This certainly facilitated the counting.

Fifteen minutes and twenty-five heights later we have tabulated the children's heights in straw units. This is a noisy and active lesson, as are many first experience lessons.

Math
Beryl S. Cochran 1965-6

MEASUREMENT
Suggestions for Other Introductory Measurement Lessons

All measurement lessons run in a similar vein. The teacher decides upon the measurement concept she wants to work on, she has the proper materials on hand, and guides the children into estimation, experimentation, and tabulation of the results.

Questions we might build a whole lesson around:

"How many of this size cube will fit into this basket?"

"Pick any linear UNIT and measure your desk."

"Make a square out of a piece of paper and measure the surface area of the work bench."

"Measure the length of a friend's arm. What kind of unit will you choose?"

"Make a graph of the length of all the girls' arms in the class."

Name ______________________________

Object to be measured ____Height of Discovery Book (h)____

Unit used	Estimation (guess)	Measurement $<$ $<$
orange rod	3	$2 < h < 3$
yellow rod	6	$5 < h < 6$
green rod	11	$9 < h < 10$
white rod	30	$27 < h < 28$
dark green rod	6	$4 < h < 5$
red rod		

Two orange rods are less than the height of "Discovery," and the height of "Discovery" is less than three orange rods.

Class ____________________

Object to be measured ____________________

Unit used	Estimation (guess)	Measurement (between)	Student's name

We now start our measurement booklet, and after each measurement lesson we fill in our recorded data:

Name ________________________________

Object to be measured ________________________________

Unit used	Estimation (guess)	Measurement (between)

NOTES FOR A CHAPTER

ON

CHILDREN LIKE SHAPES

Teacher Training Notes
Children like SHAPES
Beryl S. Cochran, 1966-67

In a FIRST GRADE class the children had forty minutes a week to work on a "free math activity."

The materials had been presented to the total class earlier, and they had experimented with them enough to know something about them.

The teacher would seldom have to give more direction than this to motivate the group:

Who

wants

to

work

with

the

Geo-Boards

Plastic Triangles

Blocks

Notes of an Elementary Math Specialist
Beryl S. Cochran, 1966-67

A Lesson with

GEO-BOARDS
and
DOT PAPER

The children have been playing freely with geo-boards for several months. I make an assignment:

> "Some time during the day I want each of you to go to the Math Activity table and make a shape on the geo-board, then draw it on dot paper, and put it in the basket where we collect the papers."

This collection will give the teacher about thirty shapes. From this collection she will be able to develop a great deal of her work in geometry. She will be able to discuss

shapes and relationships of plane figures
area
perimeter
transformations
symmetries
similarities
congruencies

Teacher Training Notes
Children Like Materials
Beryl S. Cochran, 1966-67

GEO-BOARDS
and
DOT PAPER

The young child is both artistic and creative. By making geo-boards available to him we can watch him explore geometric figures.

As he transfers his shapes to dot paper, he is gaining experience in visual representations. We can take examples of his own work to strengthen and develop geometric concepts.

Geo-boards are made from any common piece of wood with small, round-headed upholstery tacks placed two inches apart, five to a row, five to a column.

Dot-paper is an exact representation of the geo-board; that is, dots are placed two inches apart. This paper is easily run off on the mimeograph machine.

Geo-Board -- First Lesson
Y.O.U. Program, Bridgeport,
August 15-24, 1966

Questions	Commentary for the teachers
First 10 minutes	
Here is a nail board and 2 rubber bands; would you make some different shapes for me?	The children will probably make a variety of shapes. They will or are apt to make a triangular shape and call it a tent, a rectangular shape and call it a swimming pool. It is very interesting as they do this to listen to the children and take notes on their comments. It will be interesting to discuss this 10 minutes of the lesson from the children's point of view. What sort of shapes did they make and how did they perceive them.
Next 10 minutes	
Let's try something else now. Let's just take 1 rubber band and stretch it around four nails. How many different shapes can you get?	The teacher should probably have available to her dot paper if the children want to put their pictures on the dot paper. It will again be interesting to note if you think they are learning from each other. We are obviously beginning to classify four-sided figures. Take down any comments they make concerning the figures. You can either name them or not name them, but I feel very

strongly about not making this a vocabulary lesson. What we are looking for is: "Do the children look at the figures differently? Will they themselves realize the difference between the square and the rectangle, the quadrilateral, the rhombus, etc."?

Last 10 minutes

I will take a rubber band and stretch it around these four nails, and I will call this inside area ONE UNIT. Can you show me a figure that has an area of 2 units?

We will leave to the discretion of the teacher how far she pushes this; since it is a very active sort of time, you can discuss the different shapes that you get from an area of 2, or you can try to move in on the concept of area. I really feel strongly that it should be left to the teacher's discretion on how she feels her group of children are working.

Geo-Boards -- Second Lesson
Y.O.U. Program, Bridgeport,
August 15-24, 1966

Questions	Commentary for the teachers
First 10 minutes	
Here is some dot paper and here is your geo-board. Would you make an interesting shape and find the area within it?	
Next 10 minutes	
Make a shape with six vertices.	You will note that I have used the word vertices probably for the first time. If the children ask what it is, discuss it. It is, needless to say, the corners of the figure, and I would make no issue of this at all. I tend to use it very easily, and the children pick it up easily. I rather prefer using the proper term with the children, but, again, I do not want to make a vocabulary lesson out of this. You will, of course, have many different shapes. You can now discuss three things with the children individually. You can discuss the area within the figure. You can discuss the figure itself: the properties of the figure, that is, sides,

vertices, and even what it might look like to the children (the tent, swimming pool idea).

What we are most anxious to do is have the children talk. Have them be verbal and communicate to us in their natural language. It is from listening to the children and observing what they are doing that we will be able to get any feel for how they are thinking. We can then later get them to structure their work more formally.

Last 10 minutes

Here are lots of rubber bands. Make whatever shapes and designs you would like to. Here also is some dot paper, if you would like to transfer your designs to the paper.

I think it advisable to give the children freedom within a structure and again to observe the type of designs they make and the things they seem to like to do with it. I would like to collect as many of the papers that the children make as possible; however, this is not mandatory. Some of the children do like to take their work home, and that is certainly understandable; but I would like as many notes on the children in your groups as possible, and any papers that they might give you.

First Grade Math
Geo-Boards
A Specific Lesson

We handed out geo-boards and one rubber band.

> (The class had had free math activity period and some teacher-directed lessons with both geo-boards and dot paper.)

The only direction we gave was:

> "I'd like you to make some three-sided figures, four-sided figures, or five-sided figures."

A child asked,

> "MAY THEY BE CONGRUENT?"

> "Yes."

Another child said,

> "I'LL MAKE SOME SYMMETRICAL DESIGNS."

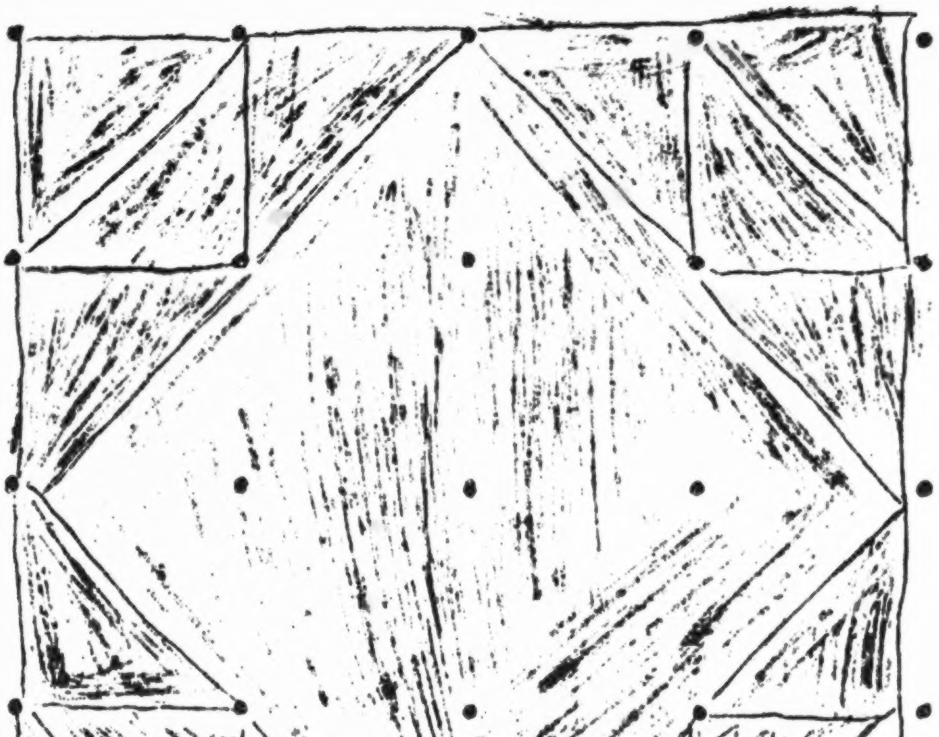
> (They had discussed congruency and symmetry in connection with other geo-board lessons.)

christine 3/10/67

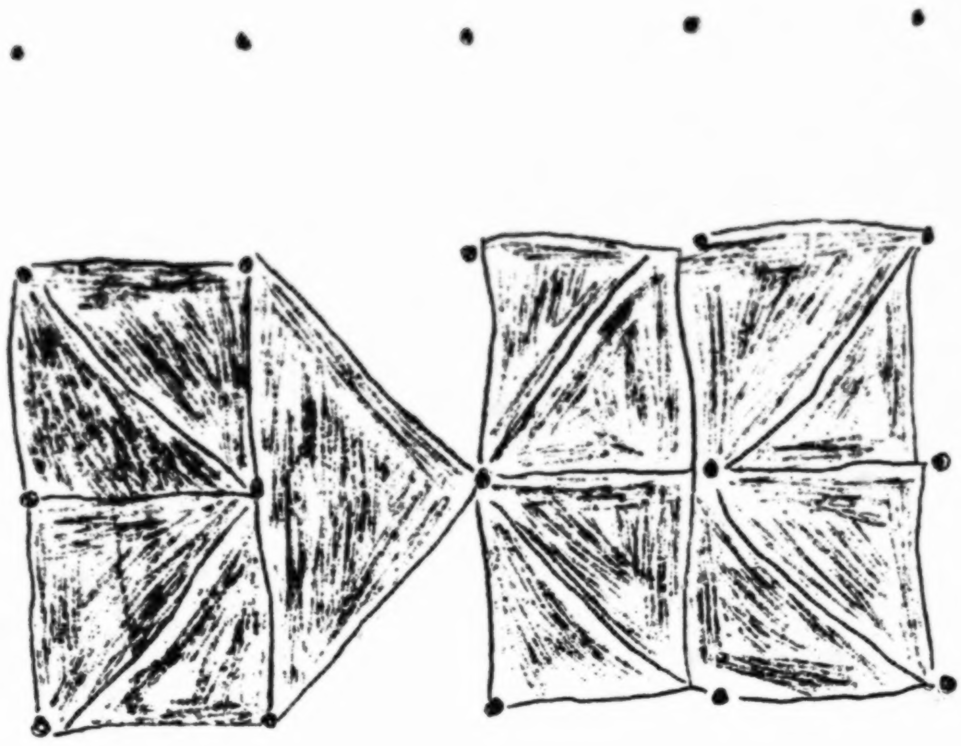
3 Sided

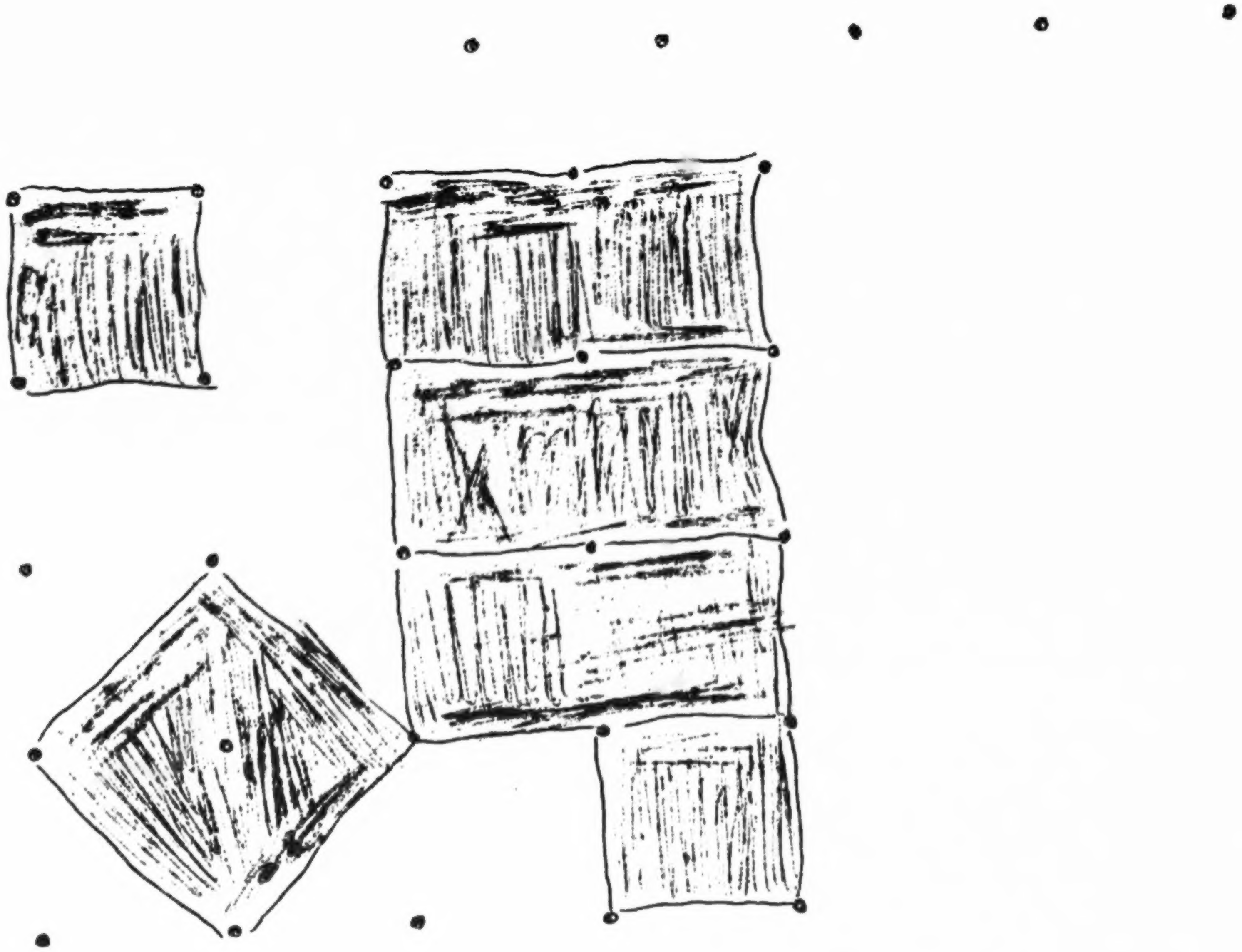
"IT ISN'T THREE-SIDED IN THE CENTER --

BUT I COLORED IN THAT WAY FIRST."



Beth 3/10/67





Mark 3/10/6

3 sided figures

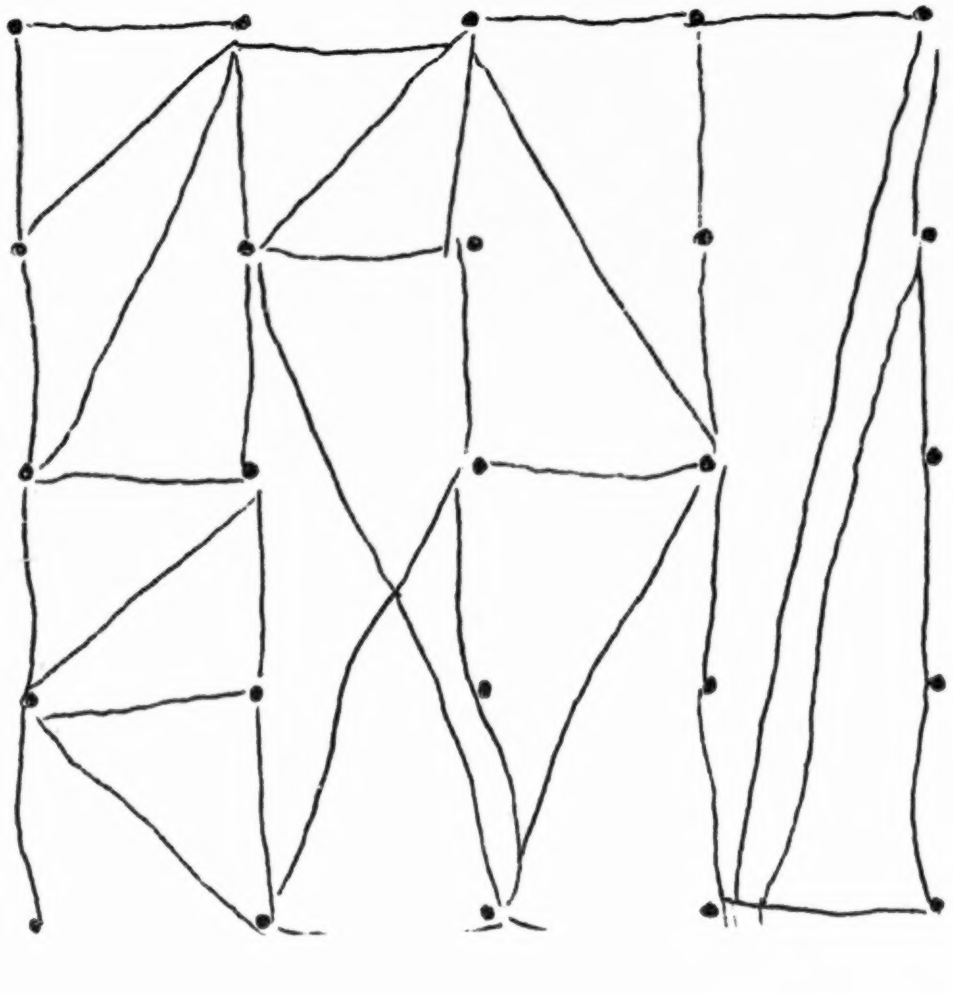
t

Congruent

Bright --

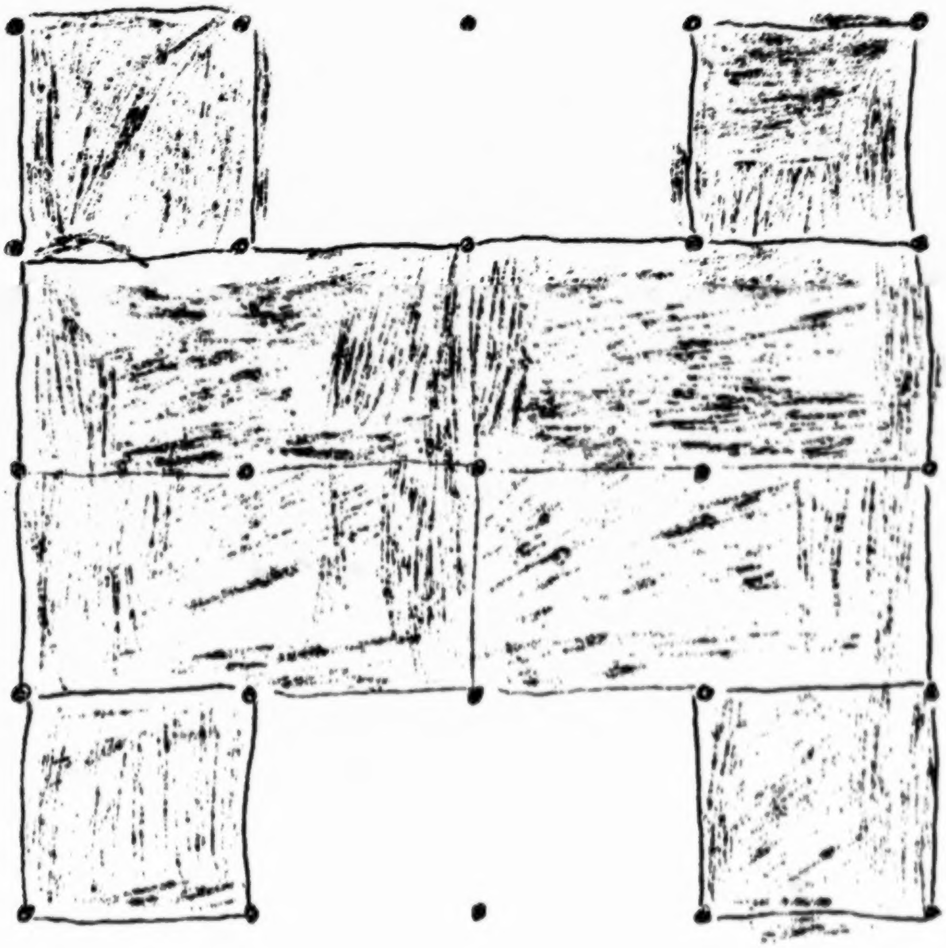
Didn't want to take time to use nail-board --

Mark was only interested in making many three-sided figures and some congruent triangles.



Berry

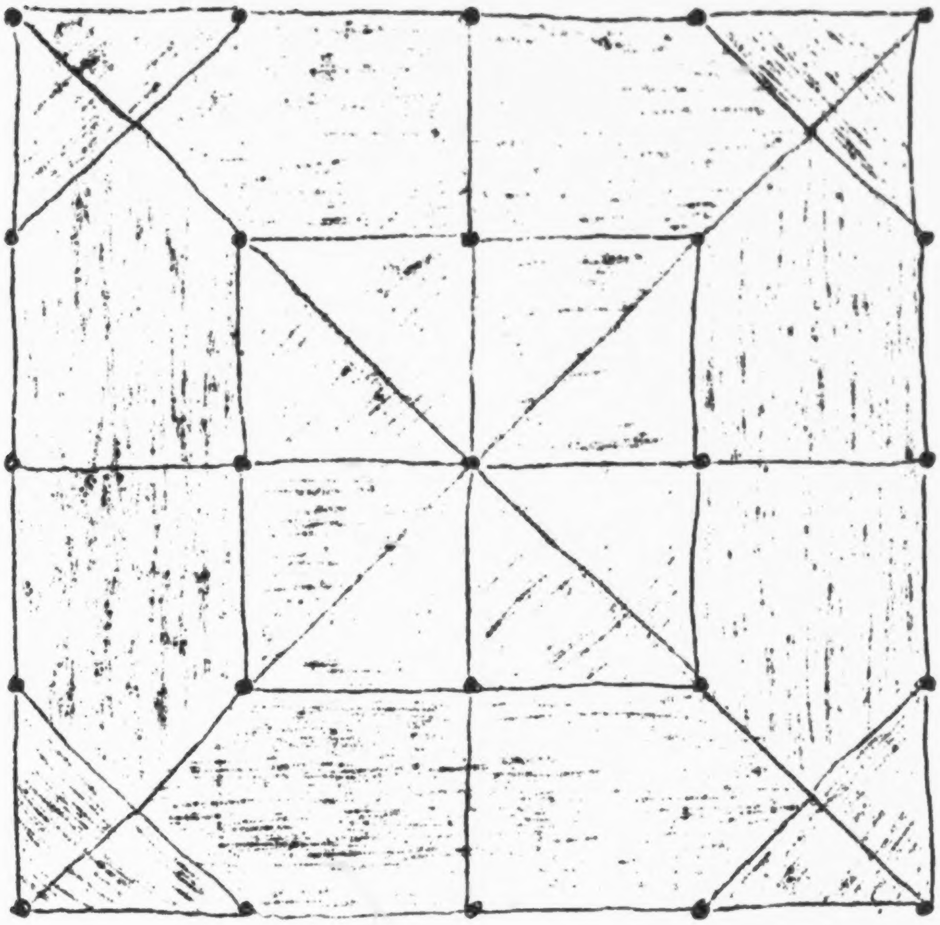
4 Sided



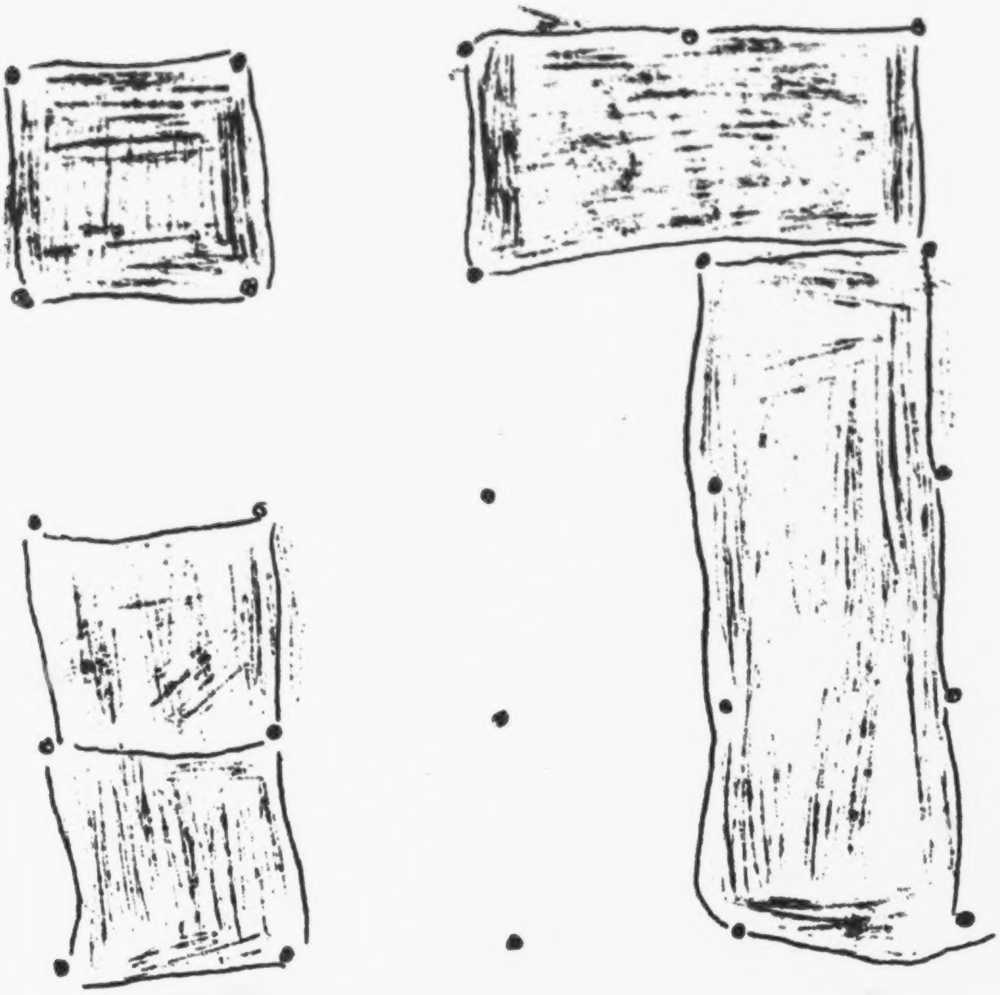
David 3/10/67

David made his own nail-board. The ones we have at school are old and some nails are loose and falling out.

He is so neat, clean, and precise -- and his work shows it.

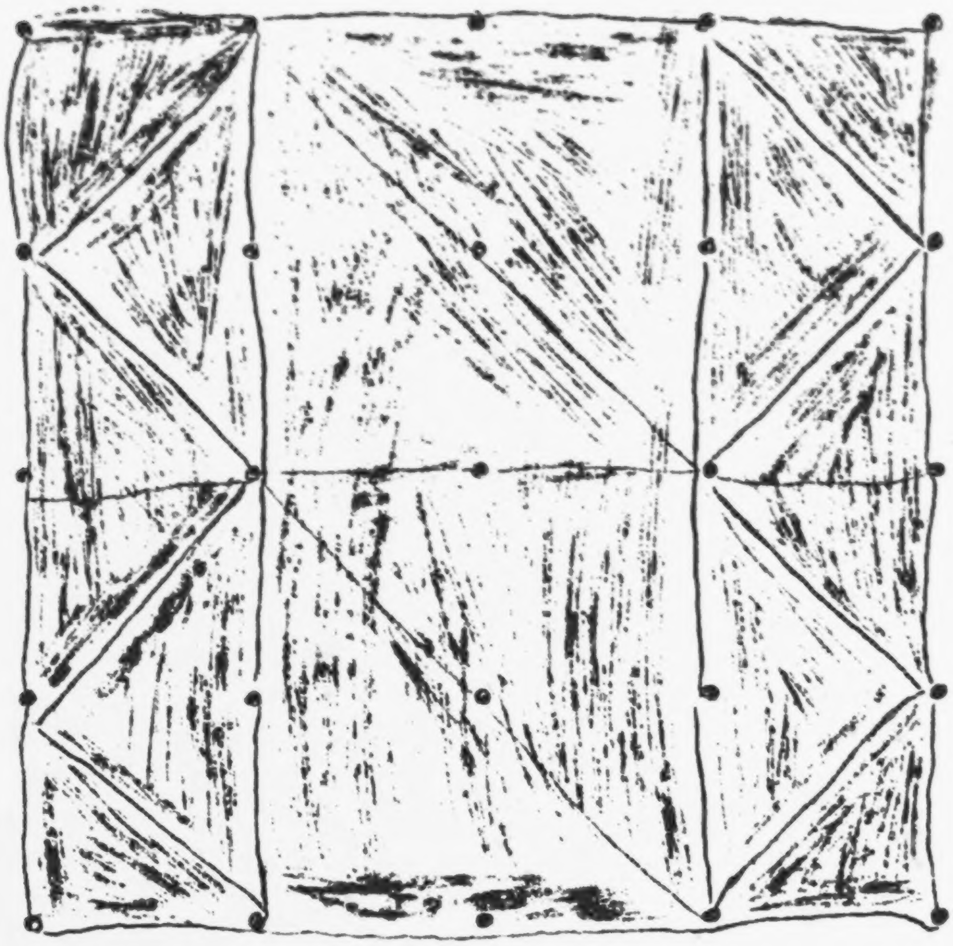


John 3\10\67
4-Sided



Berry 3\10\67

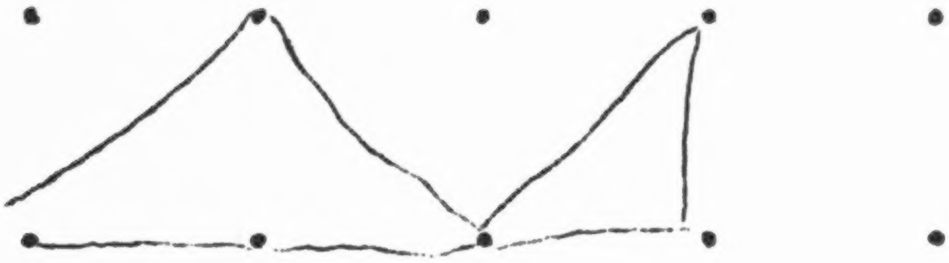
Symmetrical Dusine

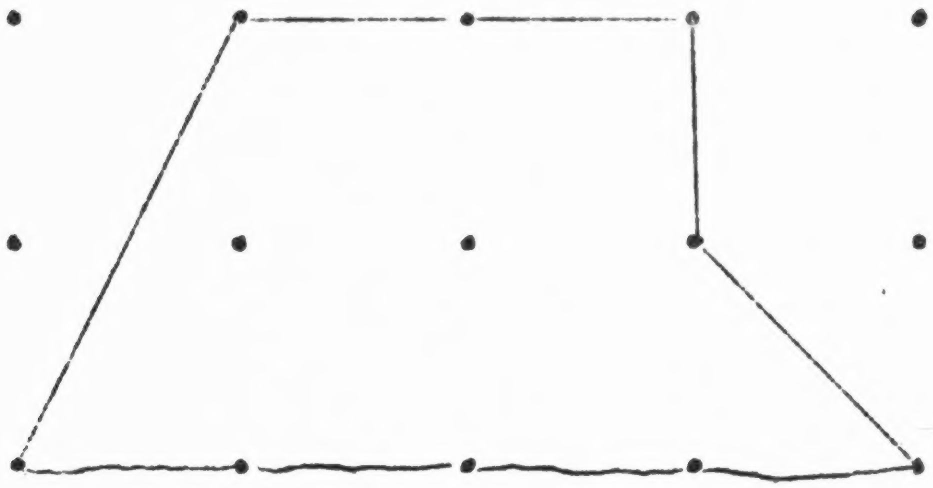


Mark Moore

Mark has only been in the class a week.

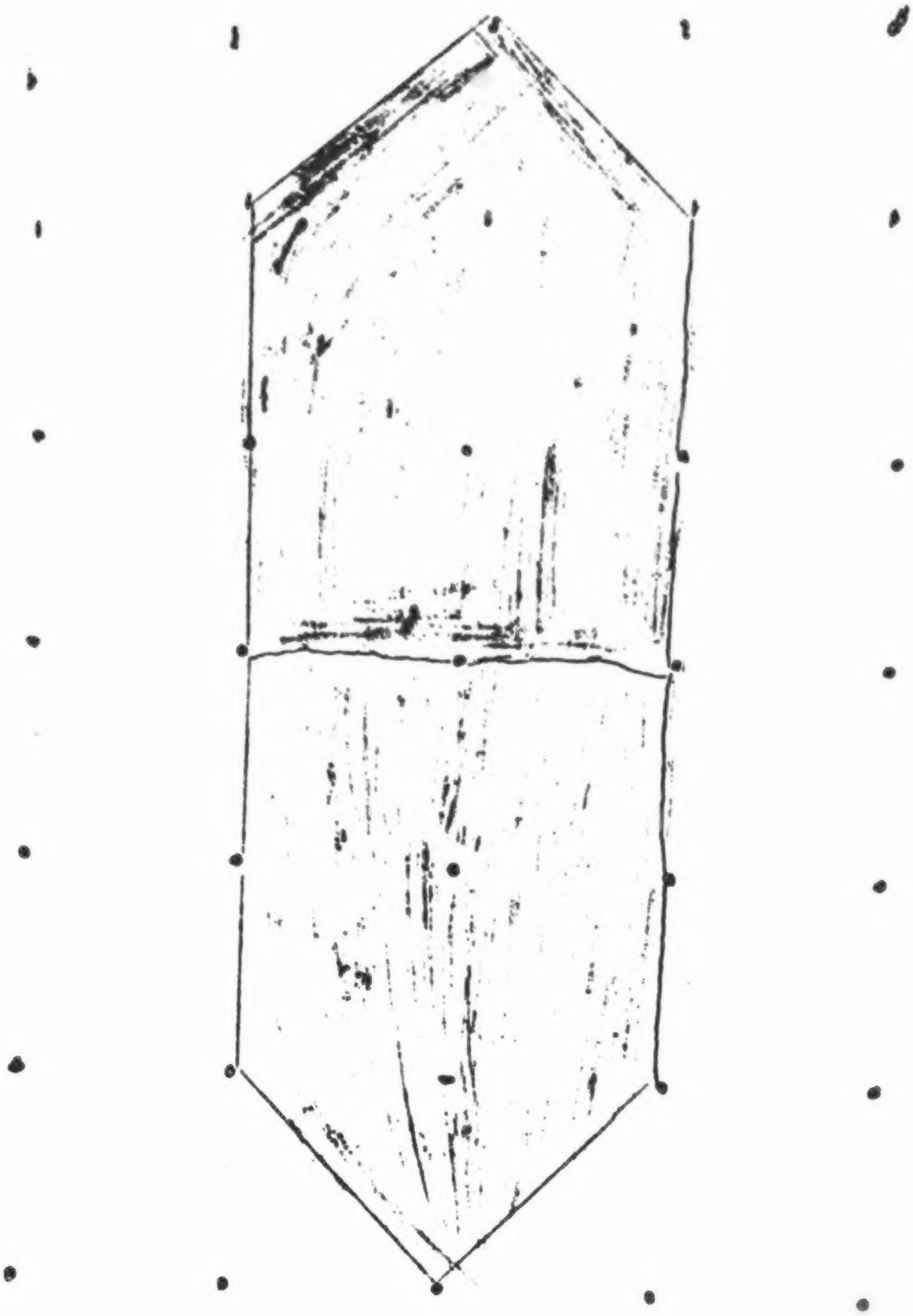
His directions were: Make a shape on your nail-board and then draw it on the dot paper.





Andy Morris 3/10/67

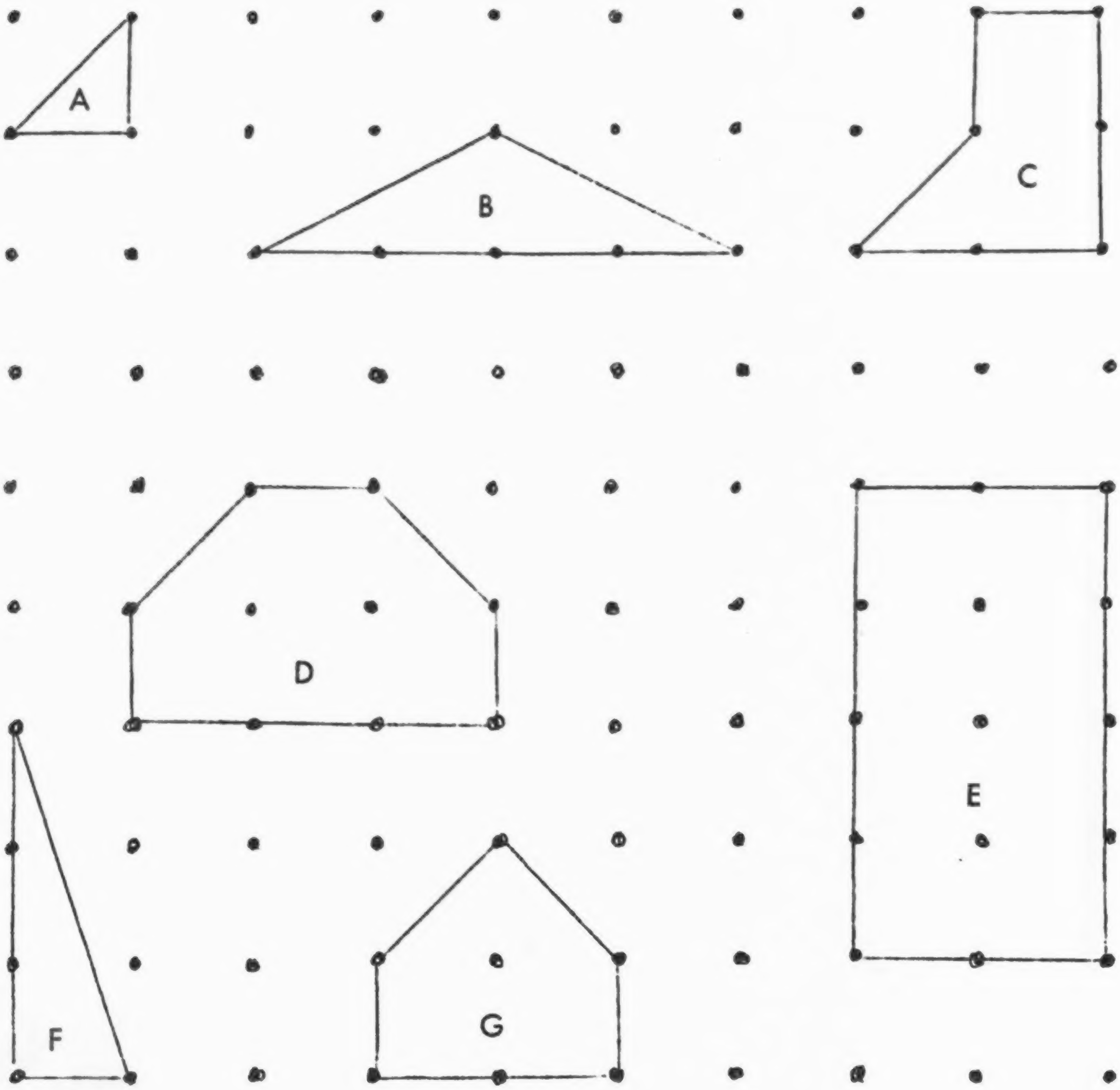
5 Sided Figures



The dot paper wasn't big enough for Andy.

"Here are some shapes you made on your nail board with one rubber band and then put on 2" square dot paper."

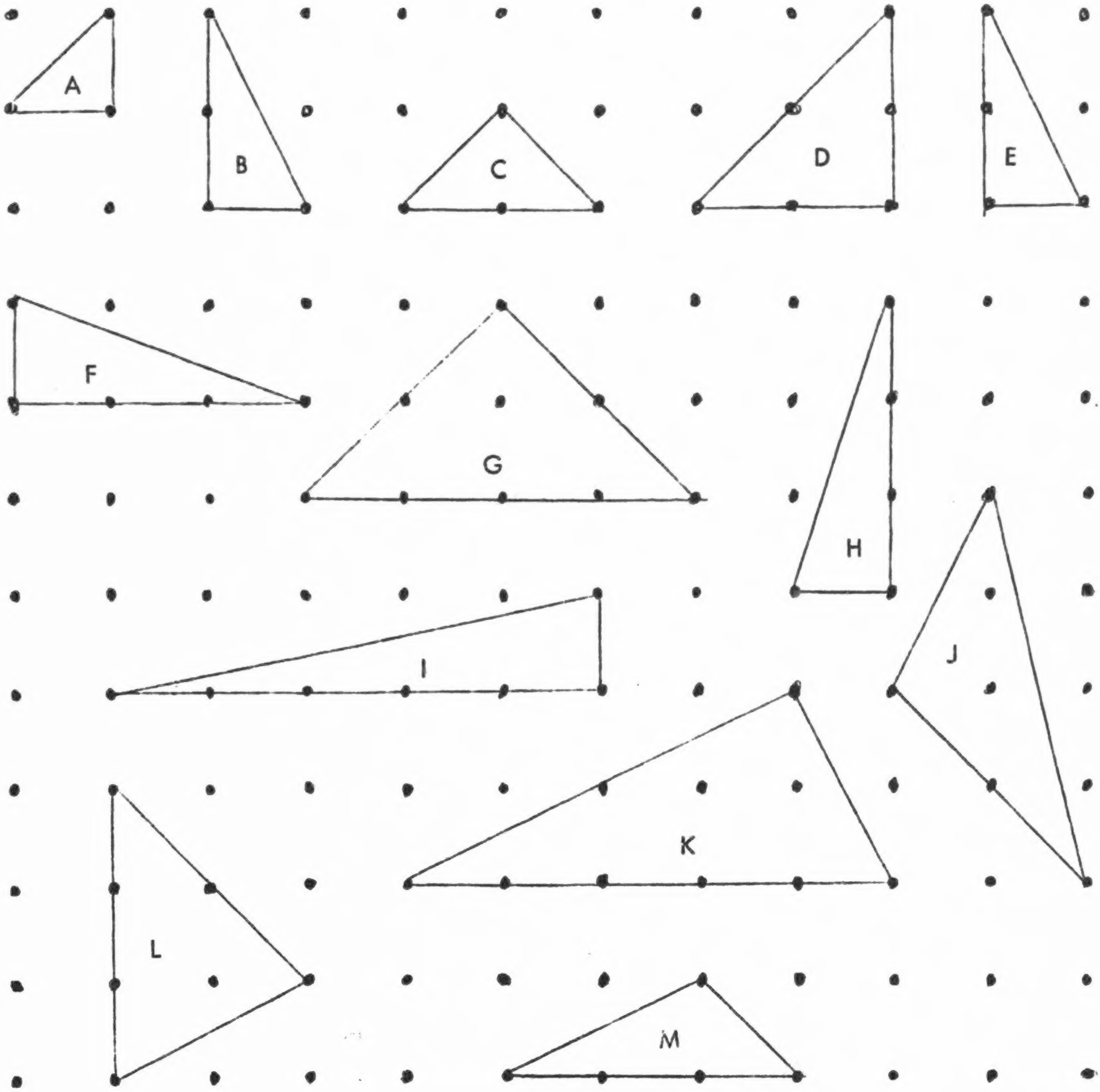
"Let's talk about them a little.



Teacher Training
Beryl S. Cochran, 1966-67

Activity Sheet

Let's talk about these shapes (1)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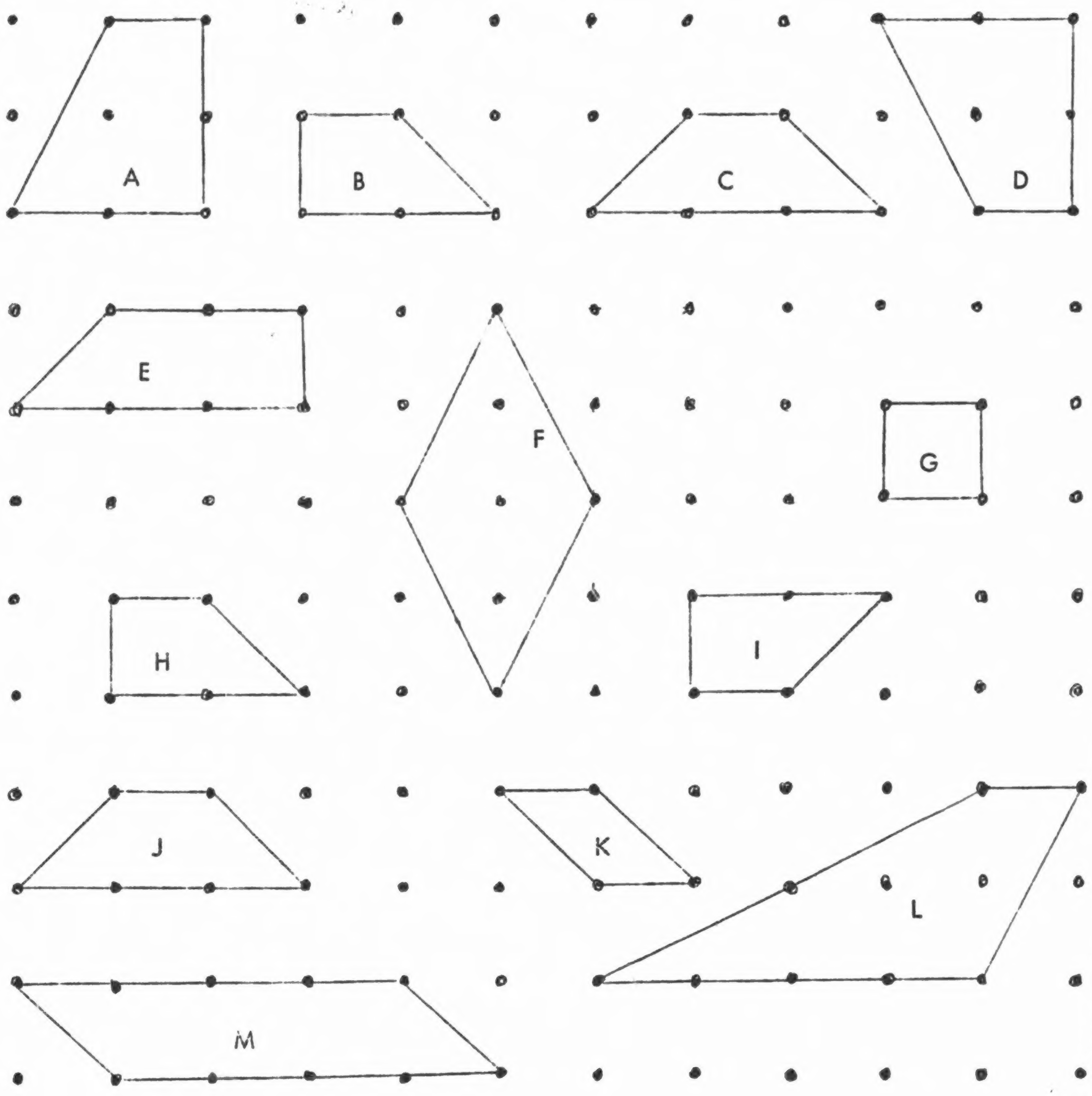
Activity Sheet

Let's talk about these shapes (2)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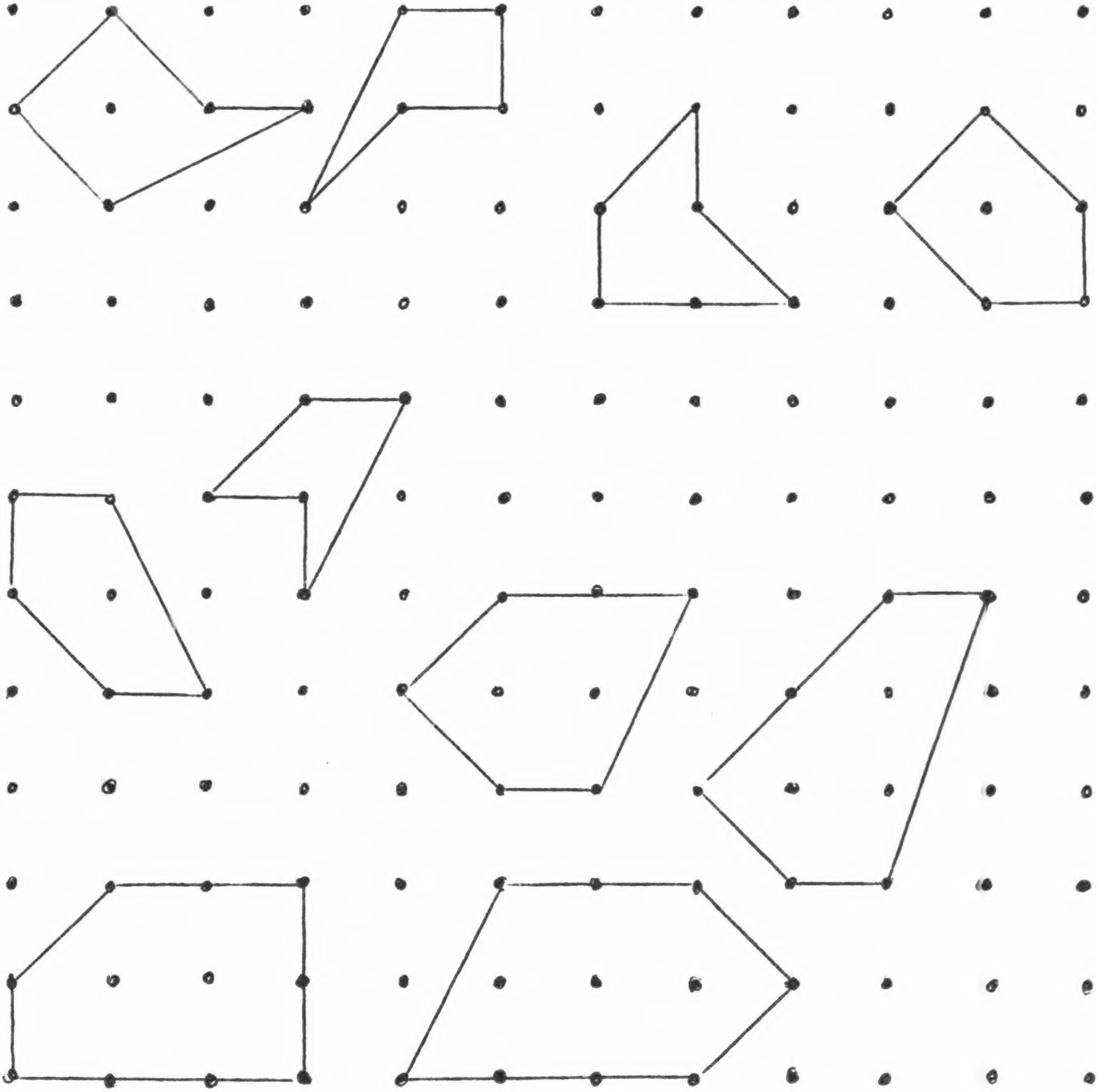
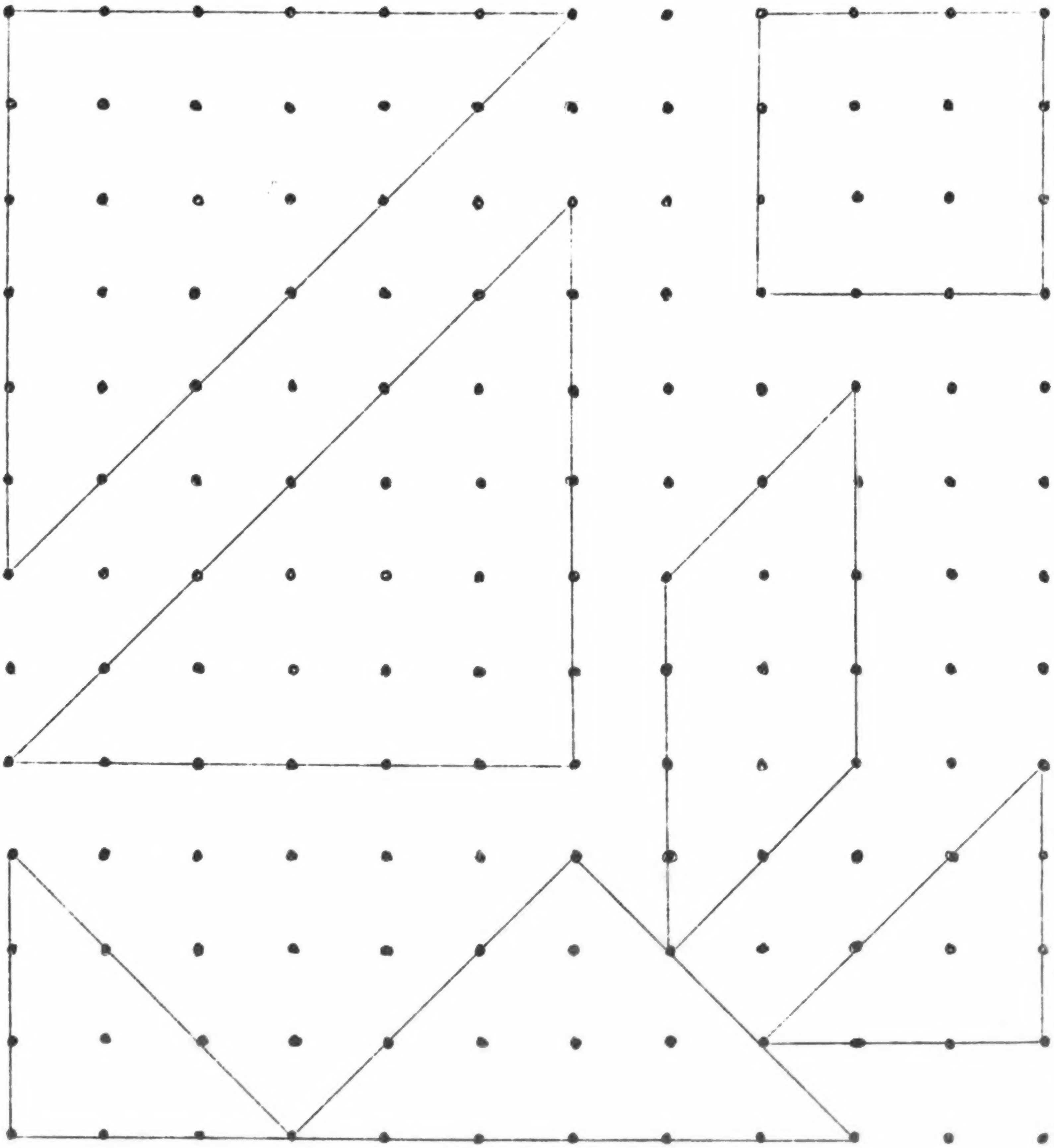
Activity Sheet

Let's talk about these shapes (3)



Children Like Materials
Shapes
Beryl S. Cochran, 1966-67



How many three cornered figures?

How many four cornered figures?

Cut out these shapes and make some designs.

This completes Mrs. Cochran's notes. On the following pages we reproduce a letter received from Miss Helene Silverman, which was discussed earlier in Appendix E.

98-20 62 Drive
Rego Park, New York 11374

October 16, 1967

Dr. Robert Davis
Director Madison Project
Big Bend Blvd.
St. Louis, Missouri

Dear Dr. Davis,

I am very excited about the unit that my class is involved in based upon the work with Odd and Even shown to the participants in the Madison Project Workshop by Edith Biggs. After teaching the concept of odd and even to my second grade class, I decided to use it as a means to get over the rut created by teaching the same social studies units in kindergarten, first and second grade.

I sent committees of children to the different rooms in my school to record the numerals on the doors. The children immediately noticed that all nine rooms had odd numbers. Conveniently, our room, 203, is the middle room on the second floor. Therefore it was easy to relate all other rooms to ours. The children noticed: 1) all the rooms ended in 1, 2, 3; 2) all the rooms on the first floor began with 1, the second floor with 2, the third floor with 3; 3) all the rooms were on the same side of the hall; and 4) that rooms ending in the same digit were in the same line.

The next day, for review, we planned an imaginary fourth floor. I asked the children why they thought that there were no rooms ending in even digits. The reasoning included: 1) our school is an odd number (P.S. 51, Queens); 2) the builder liked odd numbers; and 3) they wanted windows in the halls.

To test theory one, I asked the children who had older brothers and sisters in two neighboring schools to find out if they had odd or even numbered rooms. It was reasoned that if this was true, the two even numbered junior highs would have even rooms. (They didn't.) Since we couldn't find out about the builder, I asked the children to find out if our school was always so small. Some parents had attended the school before the rest of the building (with the even numbers) had been destroyed. (Later in the unit we will discuss the history of the building.) The children decided that all the even numbers must have been on the part that forms the tar part of the school yard.

Besides the usual discussion of the type of buildings on our school block, I'm developing a map unit based on the numbering of houses. We walked from the school gate to the corner, noting the addresses. The children again noted that all the houses ended in odd digits. But comically, they reasoned that houses in-between had been removed to make gardens. We are holding this question open until we finish investigating the street numbers. I don't know exactly where the unit will lead, but the children have become very interested in how numbers on houses increase as you get to the corner. They all want to go to their own block to see if it is odd or even, if it increases or decreases, and if anyone else knows about their discovery.

Thank you for bringing the people from England to participate in the workshop program. This is only one of many ideas that are developing in my class. How may I get in touch with these people or find out more about their work?

Thank you for your time.

Sincerely,

Helene Silverman
Teacher 2-1
P.S. 51, Queens

6 neg

Appendix F

Examples of Some Prototype Lessons That Have Not Survived "Viability" Tests.

Curriculum evolution projects in general have been accused -- perhaps unjustly -- of "not reporting their failures." Anyone who has observed many schools lately will have serious doubts about such an accusation. As, for example, the Smith and Geoffrey report cited in the main body of this report makes clear, it would be hard to create a less favorable learning environment than that which presently exists in most schools today. Cf. also the reports by Schrag (86) and by deCharms and Carpenter (31), or, for that matter, Bel Kaufman's *Up the Down Staircase* (Kaufman (58)).

Nonetheless, however necessary progress may be, the Madison Project would argue that actual progress thusfar is nowhere near sufficient. In that sense, *all* programs, texts, courses of study, and prototype lessons are "failures." None is nearly so good as it needs to be. The Project would claim -- and would cite the lessons recorded on film or audiotape as evidence -- that its prototype lessons and courses of study do represent a highly conspicuous improvement.

All of this is, however, partly beside the point. The fact is that some Project prototype lessons have proved highly "viable" or "stable," in the sense that many teachers can use them to good effect in a wide variety of settings and with a considerable diversity of students. Other prototype lessons have not possessed this "stability" or "viability," especially in the case of "Curriculum ϵ " (cf. Appendix G).

Actual viability trials have been used with much of "Curriculum ϵ ," and much of it is *not* stable.

No actual viability tests have been tried with Mrs. Machtinger's sophisticated primary-grade materials, labelled "Curriculum γ_2 " in Appendix G, but it seems quite safe to predict that much of it would *not* survive viability testing. (Cf., for example, the sequence numbered 21, 22, and 23 *for primary grades* on pages 11 and 12 of the 1965 USOE report, which culminates with children making careful and correct conjectures and axiomatic proofs for theorems about "Odd" and "Even" numbers, based on the axioms:

i) $1 + 1 = 2$

ii) 2 is even

iii) even + 1 = odd

iv) every odd = even + 1

v) even + even = even,

plus the commutative and associative laws, and a law of "excluded middle" which the children use spontaneously and implicitly. All of this does work very well indeed in good educational settings, and with Mrs. Machtinger herself serving as teacher. It would not seem advisable to propagate this program widely at present. Of course, Mrs. Machtinger seems to have tapped an ability of quite young children that has previously gone unrecognized. Patrick Suppes has conducted correspondingly sophisticated mathematics sequences successfully with primary graders. Conceivably such early instruction may change a child's subsequent thinking quite markedly. This, too, deserves study, but the Madison Project's increasing involvement with many thousands of teachers in the cities of Chicago, New York City, Los Angeles, and Philadelphia, and in San Diego County, have forced the Project to become increasingly concerned with "viability" or "stability."

We wish to give here two specific portions of "Curriculum ϵ" that have conspicuously failed to meet the criterion of stability. In both cases the goal has not been abandoned, but alternative paths to it have been found.

Example 1. Finding matrix inverses: e.g., if $\underline{A}$ is a 2-by-2 matrix, and if $\underline{I}$ is the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

find a matrix A^{-1} such that

$$\underline{A} \times \underline{A}^{-1} = \underline{I} .$$

This was pursued in the academic year 1960-1961, with "Class A" (cf. Appendix C), when they were sixth graders, by attempting to follow this path:

i) Deal with examples such as

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 10 \\ 21 \end{pmatrix},$$

and interpret this algebraically only, as a "transformation" or "mapping" with an "input" and an "output"

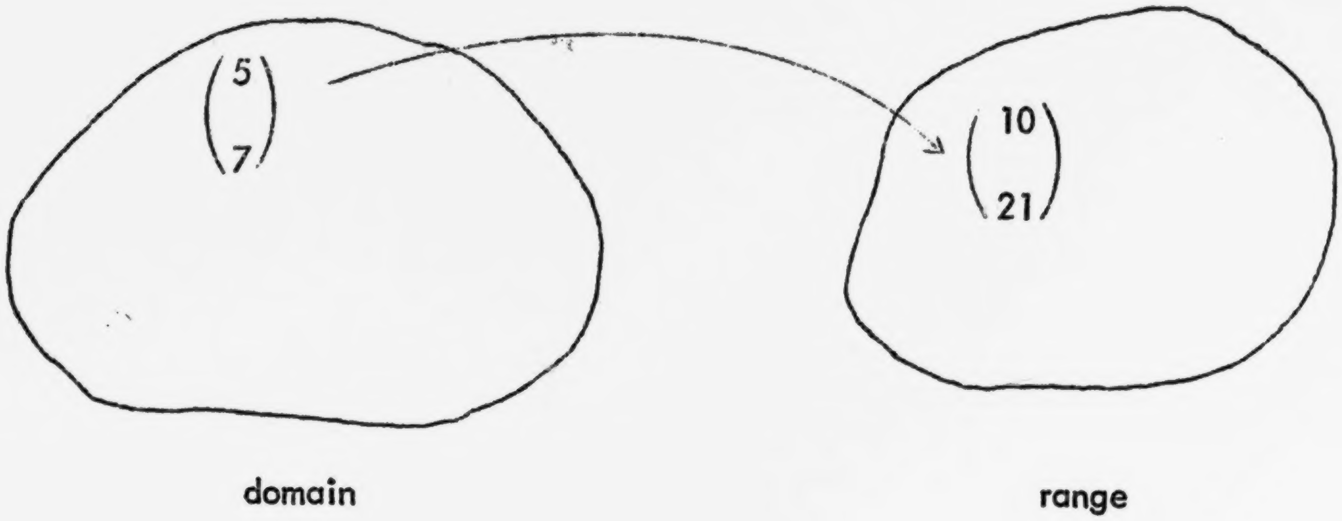
$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \times \underset{\substack{\uparrow \\ \text{input}}}{\begin{pmatrix} \ \\ \ \end{pmatrix}} = \underset{\substack{\uparrow \\ \text{output}}}{\begin{pmatrix} \ \\ \ \end{pmatrix}}$$

More picturesquely, regard this as an "archery contest" that "shoots" -- say --

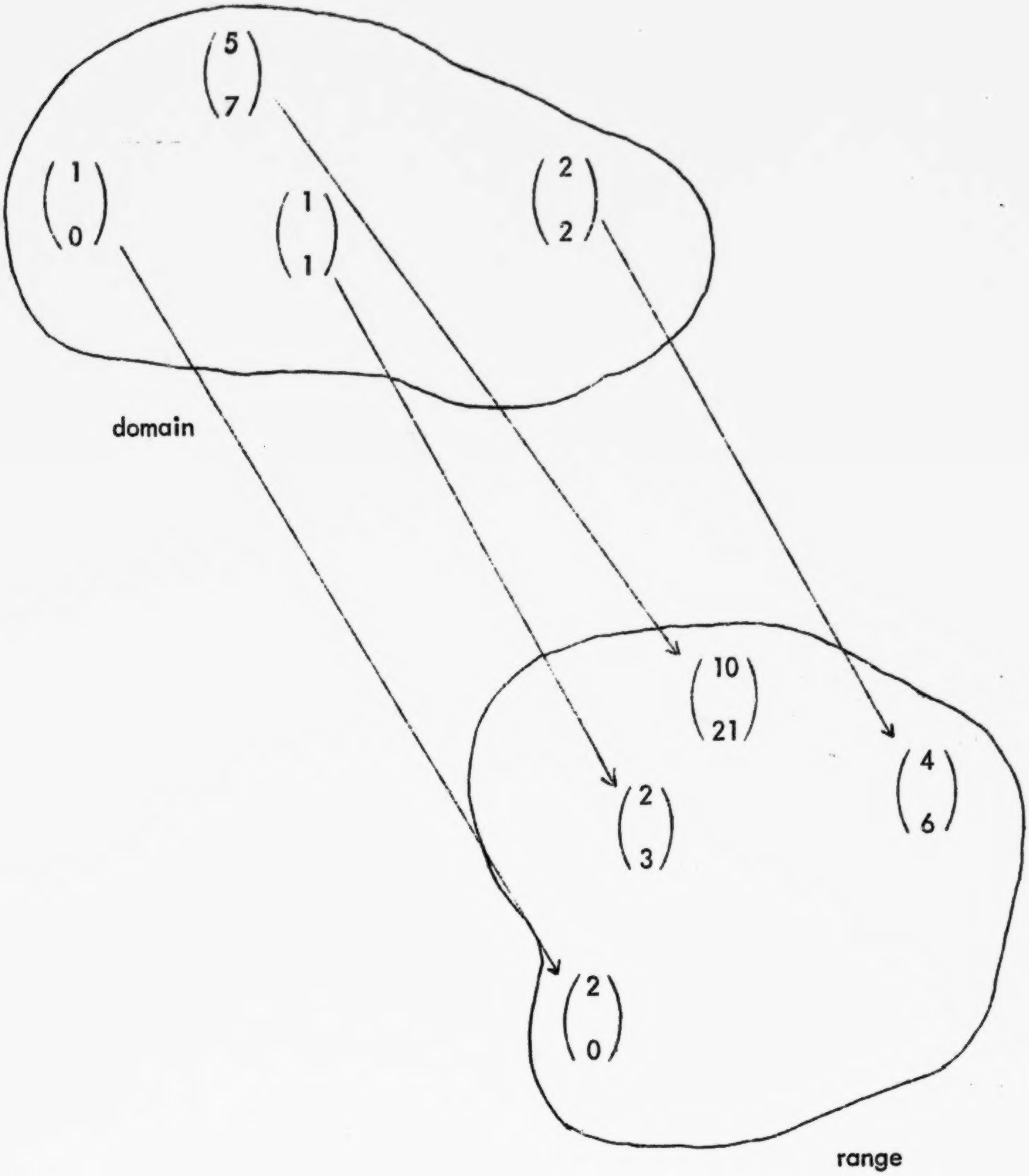
$$\begin{pmatrix} 5 \\ 7 \end{pmatrix}$$

into

$$\begin{pmatrix} 10 \\ 21 \end{pmatrix} \quad :$$



One can extend this to consider many more "shots":



$\begin{pmatrix}5\\7\end{pmatrix}$
$\begin{pmatrix}1\\0\end{pmatrix}$
$\begin{pmatrix}1\\1\end{pmatrix}$
$\begin{pmatrix}2\\2\end{pmatrix}$
domain
$\begin{pmatrix}10\\21\end{pmatrix}$
$\begin{pmatrix}4\\6\end{pmatrix}$
$\begin{pmatrix}2\\3\end{pmatrix}$
$\begin{pmatrix}2\\0\end{pmatrix}$
range

ii) Introduce the basis vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

iii) Note that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} A \\ C \end{pmatrix}$$

and

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} B \\ D \end{pmatrix},$$

so that a knowledge of the image of

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

would determine the first column of the "shooting" matrix, and, correspondingly, a knowledge of the image of

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

would determine the second column of the "shooting" matrix.

iv) Interpret the inverse matrix in terms of an "inverse archery contest," where, when we see an arrow in the target, we ask which archer shot it (purists may argue that the metaphoric language here is somewhat inexact, but it is doubtful that that is the main difficulty; probably the main difficulty is the large number of independent concepts that must be kept in mind and related to each other; this task involves "chaining" concepts one onto the next, rather like making protein molecules, and the "chain" becomes quite long indeed before we reach our goal; undoubtedly one could teach this to sixth graders, but the Project is concerned especially with whether this does, or does not, "feel" like a natural thing for sixth-grade children to be doing).

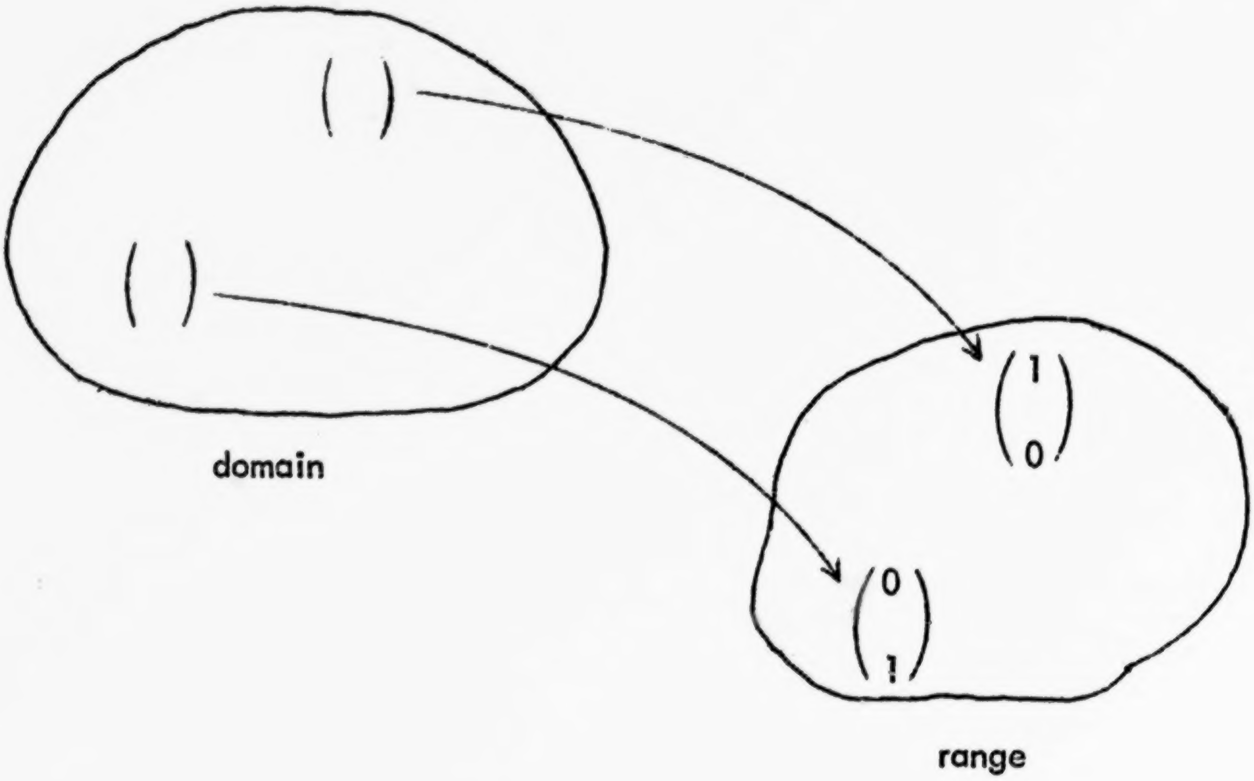
v) Then what we are seeking for A^{-1} is the "shooting matrix" for the inverse archery contest. If, then, we obtained

$$\begin{pmatrix}1\\0\end{pmatrix}$$

and

$$\begin{pmatrix}0\\1\end{pmatrix}$$

as image vectors in the range, and knew what vectors in the domain corresponded to these two vectors, our problem would be solved, for we should then know that


( )
( )
domain
$\begin{pmatrix}1\\0\end{pmatrix}$
$\begin{pmatrix}0\\1\end{pmatrix}$
range

the two indicated vectors in the domain would be the two columns of $\underline{A^{-1}}$, by property iii, above.

vi) But can we get

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

in the range? Yes, by using the properties -- which we observe by generalizing from several instances -- that, for any linear transformation T, any scalar α, and any vectors $\vec{u}$ and $\vec{v}$, we must have:

V': $T(\vec{u} + \vec{v}) = T\vec{u} + T\vec{v}$

V'': $T\alpha\vec{u} = \alpha T\vec{u}$.

Consequently, by adding or subtracting we can create a zero as one entry in a non-zero vector; thereafter by multiplying or dividing we can make the other entry one. We need only be careful of our bookkeeping, and make sure that whatever operations we use on vectors in the range, we do the same operations on their inverse images back in the domain.

In a relatively brief presentation to Class A when they were beginning sixth grade, we tried this approach. One student, whose nametag reads "Debby H.," fully understood and could explain the relationship of each piece in the chain to every other piece.

This is the kind of situation which often confronts a curriculum project. It goes far to explain why the conventional wisdom of educational research usually isn't much help. There are, quite literally, so many millions of decisions to make that one could not settle them by elaborate psychometrics.

In this case, to name only a few possible next steps:

i) We could find students for whom this approach was satisfactory (remembering that Debby H. had not been "drilled" on this, but was able to operate effectively with her own original creative synthesis of these various ingredients). We could do this by looking for an appropriate I.Q. range (Debby H.'s I.Q., 140+, was the highest measured I.Q. in the class), or for appropriate cultural background (probably only to find that this was largely irrelevant), or for an appropriate grade level. (After all, in the 1940's this approach was a standard tool for graduate students majoring in mathematics.)

ii) We could retain this approach, but build each subsidiary skill or concept more slowly and firmly before moving on to the next. (It is our intuitive judgement that sixth graders would find a slow, careful development of this quite uninteresting. However, as David Hawkins has pointed out, you cannot test very many alternatives with actual classes of children. Any significant trial usually involves protracted and expensive advance preparation-- we had invested the academic year 1959-1960 in "preparing" Class A, in getting them ready for this kind of content. It would take another year to prepare another class -- and there are, quite literally, millions of decisions which a curriculum evolution project must make.)

iii) We could keep the sequence and pacing the same, but change the style of classroom presentation. (This always helps some -- here you are dealing with the art of achieving a better performance.)

iv) We could discard this topic as unsuitable.

v) We could tentatively retain the topic, but try to develop a quite different approach to it. This, in fact, is what was actually done, as follows:

Revised Approach to Matrix Inverses. The Project shifted the grade level, from grade six to grade nine (largely because we did not want to deal with 140+ I.Q. students, but with, perhaps, 115+ I.Q. students), and also devised a quite different line of attack, as follows:

i) The children received extensive experience in working with matrices in various ways before the subject of matrix inverses was raised;

ii) The children also learned something about determinants;

iii) The children were asked to compute the matrix product

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & {}^{-}4 \\ {}^{-}2 & 3 \end{pmatrix} = \quad ?$$

iv) The question of finding matrix inverses was then raised. It was, obviously, noticed that we already knew the multiplicative inverse of the matrix

$$\begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix},$$

and by direct calculation

$$\begin{pmatrix} 3 & {}^{-}4 \\ {}^{-}2 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix} =$$

it was confirmed that, surprisingly, we also knew the the multiplicative inverse for the matrix

$$\begin{pmatrix} 3 & {}^{-}4 \\ {}^{-}2 & 3 \end{pmatrix}.$$

v) The children were then asked to try to find the inverse for

$$\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix},$$

and also to test this atypical commutativity that had been observed in the previous example.

vi) One then went on to a sequence of other problems, in every case where the determinant was one, although this fact had of course not been mentioned to the students!

vii) Sooner or later the teacher invokes the procedure which the Project refers to as "Torpedoing" (cf. Davis (21)); that is to say, the students are asked, perhaps, to find the multiplicative inverse of the matrix

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix},$$

where, for the first time, they encounter a problem with $a_{11} \neq a_{22}$. When they come to check their work, they find

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} \times \begin{pmatrix} 5 & {}^{-}2 \\ {}^{-}7 & 3 \end{pmatrix} = \begin{pmatrix} 11 & {}^{-}4 \\ 14 & {}^{-}5 \end{pmatrix},$$

which is not at all satisfactory.

If all else fails, the students can resort to a Polya-like device of trying some easier problems. For example, for

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

it is easy to find that the inverse must be

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},$$

but this may not prove too helpful. One teaching stratagem that can be used is for the teacher to see to it that the product

$$\begin{pmatrix} 8 & 3 \\ 5 & 2 \end{pmatrix} \times \begin{pmatrix} 2 & {}^{-}3 \\ {}^{-}5 & 8 \end{pmatrix} = \quad ?$$

occurs somewhere. If the students have really become involved in the search for matrix inverses, they will almost certainly see here a clue to aid their researches, and, indeed,

$$\begin{pmatrix} 3 & {}^{-}2 \\ {}^{-}7 & 5 \end{pmatrix}$$

works quite nicely as the desired inverse of

$$\begin{pmatrix} 5 & 2 \\ 7 & 3 \end{pmatrix} .$$

There has, of course, still been no mention of determinants. This provides the next opportunity for "torpedoing":

viii) The teacher presently gives the students a problem such as "find the multiplicative inverse for the matrix

$$\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} .$$

When the students come to verify their answer, they get

$$\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 4 & {}^{-}5 \\ {}^{-}2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} ,$$

which is not exactly what is wanted. But no matter! The difficulty is easily overcome; we got twice what we desired, so we ought to divide by 2; a little experimentation readily produces

$$\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & {}^{-}2.5 \\ {}^{-}1 & 1.5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

ix) The remaining task is for the teacher to vary the determinant, and see if the students can find a way to predict in advance the factor by which they must divide. (Use can again be made of relatively simple or striking examples, such as

$$\begin{pmatrix} 14 & 1 \\ 13 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 1 \\ 12 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 12 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 14 & 3 \\ 4 & 1 \end{pmatrix}$$

and so on.)

Remark: Incidentally, this parallels in a very striking way the Project's treatment of quadratic equations, where, at first, the two roots are simple unequal primes, such as in

$$(\square \times \square) - (5 \times \square) + 6 = 0$$

$$\{\ ,\ \} \quad .$$

As the students become more sophisticated, successive complexities are added: roots not prime, the use of *one* as a root, a "double" root, and finally negative or fractional roots. (Cf. Davis (20), Chapter 3.)

There has been no opportunity as yet to submit this new 9th grade treatment of matrix inverses to any extended "viability" tests. The Project will be grateful to any teachers who do use this material and who report results to the Project, preferably via audio-tape-recorded classroom lessons. The suggested level is grade nine.

Example 2. The ϵ, N definition of limit of a sequence, with 8th graders.

This version can be seen on two films with "Class A," showing actual classroom lessons. Again, "Debby H." clearly understands, but many other children in the class do not.

This first ϵ, N version was therefore dropped, and replaced by an approach via bounded monotonic sequences and nested intervals. This newer version, also, can be seen on films.

Appendix G

The Various Curricula Developed by The Madison Project

Since, as should by now be clear, the Project attempts to adjust classroom experiences to the actual children involved, it does not make use of rigidly determined "curricula" which are fixed a priori, or which remain constant as time passes. Thus, any attempt to describe the Project's diverse curricula is in fact an exercise in taxonomy.

Here is one such attempt, in which some attention is paid also to the developmental history of the Project.

Within the main body of this report five principle curricula have already been identified:

"Curriculum α": a basic "unified" curriculum for grades two through eight (or thereabouts; grade level designations are always appropriate, since the Project makes every attempt to move toward non-graded schools); purpose: to provide a foundation for combining arithmetic, algebra, geometry, and science.

"Curriculum β": the "assembled" curriculum for grades two through eight. This curriculum is presently the basis for the Madison Project's "big city" workshops in Chicago, New York City, Philadelphia, Los Angeles, and San Diego County.

It is essentially an extension of "Curriculum α," and was created as a consequence of the Project's judgement that "Curriculum α" had certain weaknesses, conspicuously these:

i) It needed more emphasis on such fundamental arithmetic topics as place-value numerals and the arithmetic of rational numbers;

ii) It needed more extensive areas of contact with science;

iii) It needed provision for more use of manipulatable physical materials;

iv) It needed more provision for individualization of instruction and for small-group work;

v) It needed to take more effective advantage of similar curriculum materials developed by other projects and by other individuals.

Consequently, the Project assembled a "larger" curriculum by creating more individualized "shoe box" projects (i.e., student projects or "activity packages" for which the necessary materials and instructions can be stored in a small package; the package can then be "checked out" of a library just as a book can be, etc.) which were mainly the work of three Project staff members, Donald Cohen, Gerald Glynn, and Louise Daffron, and by assembling and combining ideas and materials previously developed by E.D.C./ E.S.S., by the Nuffield Mathematics Project under Geoffrey Matthews' leadership, by Leonard Sealey, E.E. Biggs, Z.P. Dienes, John Trivett, David Page, Marion Walter, Lauren Woodby, Joan O'Connell Barrett, Victor Wagner, and others.

"Curriculum γ" is identified in the main body of the report as "a simplified curriculum for primary grades" -- i.e., nursery school through grade two, or so. It is mainly the work of Beryl S. Cochran, although R. B. Davis, Gerald Glynn, George Arbogast, Louise Daffron, and others, have also worked on it.

Appendix E presents "Curriculum γ" in some detail, mainly via the reproduction of some of Mrs. Cochran's actual notes for her teacher education program. Appendix E also contrasts "Curriculum γ" with three other primary grades programs which the Project has developed.

Before pursuing further the curricula emphasized earlier in this report, we go back to the beginning, start there, and proceed historically.

Curriculum No. 1. The beginning occurred at the Madison School, in Syracuse, New York, with a special program for non-achieving culturally-deprived 7th grade children.

Since all previous instruction in arithmetic had demonstrably failed with these children, it seemed exceedingly unwise to prescribe more of the same. As an alternative, it was decided to teach them a specially-created program, "modern" in spirit and notation, that was built upon selected portions of algebra, analytic ("coordinate") geometry, and the study of functions. Ideas of notation and special topics were adopted or adapted from the work of W. Warwick Sawyer, Max Beberman, David Page, and Stewart Moredock.

The 7th grade class consisted of twenty children, six girls and fourteen boys. It was considered the "lowest" group of seventh graders in the school in terms of achievement, especially in mathematics. Achievement test levels for these children ranged from grade two to grade five. The regular teacher of this class was Mrs. Jane Downing; Principal of the school was William Bowin; the Madison Project "specialist teacher" was R. B. Davis.[110]

Mrs. Downing always taught such classes on a completely individualized basis. Because of the wide spread of achievement levels, Mrs. Downing would often have a class of sixteen to twenty children where each child was working from a different textbook. Even if two children were using the same book, they would ordinarily not be working on the same chapter or topic.

The desks and chairs were bolted to the floor, in rows, facing the "front of the room," although the front, right hand side, and back walls of the room were covered by blackboards. Mrs. Downing virtually never stood at the front of the room; aside from shouting "Fire," it is hard to imagine much of anything that might appropriately have been said to all twenty children in the class; they had very little in common, except that none of them was presumed to be "very good at arithmetic." (Subsequent developments were to cast doubt on this diagnosis.) Nonetheless, their actual difficulties, and the new work each might be prepared to cope with, varied from child to child.

It was arranged that the special program would operate three days a week, on Mondays, Wednesdays, and Fridays. Mrs. Downing would teach arithmetic all five days each week, but on Mondays, Wednesdays, and Fridays the children had their choice (and they really did!) of sitting near the front of the room and working on arithmetic with Mrs. Downing, or else sitting near the rear of the room and working with the Madison Project teacher on the

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[110] Part of the present report is drawn from Mrs. Downing's notes.

"algebra" program. The special program was simply labelled "algebra" -- not, perhaps, altogether accurately. The classroom was physically large enough to make dividing up quite feasible.

Student choices were fairly stable, and the children who elected "arithmetic" mainly stuck with this choice, and similarly for those who elected "algebra." Once in a while a student would pay a visit to the other end of the room to see what, if anything, he was missing.

The "algebra" group used three main modes of communication: either students working independently, and having occasional conferences with the teacher; or else two or more students competing to see who could solve a particular set of problems first; or else a student working as "teacher's aide" and instructing other students in some particular concept or technique which he himself had already learned.[111]

One physically large boy, Harvey, who had had something of a reputation as a bully in physical fights, found and used a new route to achieving status: first he became the fastest and most proficient student on most topics and techniques; second, he became an "assistant teacher." At this stage an adult teacher heard Harvey remarking to another child: "Oh, that stuff is pretty easy to do, but it's really hard to teach it." Harvey was a new kind of hero. (Harvey's older brother, a marine, had recently studied algebra.) Harvey's physical attacks on other children were reported as decreasing at this time.

Another (much quieter) child, named Clarence, improved the quality of his work so much that he was transferred to a "better" section, with a different teacher. He then became a chronic truant until he succeeded in persuading the guidance counsellor to return him to the "bottom" section, whereupon his work continued to improve dramatically. It became clear that many of these children were, in actual fact, college capable.

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[111] Notice that little or no use was made of what, in 1967, has become the Project's method of choice for most class work: children working together in groups of two, three, four, five, or six children per group, with the group working cooperatively on a a common task.

Although this was the most effective individualization of instruction the Project has ever achieved, it is interesting to observe that this was entirely a paper-and-pencil operation. No physical materials whatsoever -- beyond books, paper, pencil, chalk, and erasers -- were ever used, either in the arithmetic program, or in the "special" program. The three-times-a-week "special" class operated for seven months, beginning (according to Mrs. Downing's notes) in November, 1956, and concluded in May, 1957.[112]

Other teachers in Syracuse were making extensive use of physical materials, notably Mrs. Doris McLennon and Walter Garner, but the significance of this was not appreciated by the Madison Project staff at that time. (On the other hand, so far as is known, no other mathematics teacher in Onondaga County completely individualized instruction as Mrs. Downing did.)

During its seven months of operation, the "special" program produced dramatic improvement in the performance of the students, which resulted in turning the Project's work in an entirely different direction.

"Curriculum 1" formed part of the basis for the contents of the two volumes:

Robert B. Davis, Discovery in Mathematics: Student Discussion Guide, Reading, Massachusetts: Addison-Wesley, 1964.

Robert B. Davis, Discovery in Mathematics: A Text for Teachers, Reading, Massachusetts: Addison-Wesley, 1964.

The route, however, is not direct. Several other tributaries joined the stream before these two volumes were completed.

One attribute of "Curriculum 1" deserves emphasis. This program made very limited use of verbal resources; the children in the Madison School class were not verbally proficient. Reaching them effectively resembled somewhat the efforts to signal an intelligence in outer space -- very little of the communication can be transacted in spoken English. Something more basic must be sought for. In the case of outer space, recourse is being had to the sequence of prime numbers, signaled in digital fashion, etc. In the case of the seventh grade children, the Project sought a more fundamental resource in the device of *first* sharing a common experience with the students, and *then* building on this common experience. This was, for the Project, the beginning of its "*do* ... then *discuss*" strategy of teaching. It, or variants of it, nowadays go under such names as "discovery teaching" or "teaching in the inductive mode." This methodology is discussed extensively in certain circles. Nonetheless, virtually no textbooks show any influence of this approach, even in 1967,[113] and any such instructional strategy can be witnessed in exceedingly few classrooms in the United States today. It has not swept the nation's schools, nor its colleges.

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[112] Various Project records are not in complete agreement on these dates.

The decision to minimize the use of language is based upon the idea that the student's internal data processing equipment can cope with mathematics itself, but his input-output equipment cannot effectively process communications in the English language. This decision has a number of corollaries besides the "do ... then discuss" strategy of teaching. For one thing, it implies that instructions and explanations either will not be given at all, or else they will be kept as simple as possible, and usually will be kept on an *informal* level linguistically.

This, in turn, implies that notations, wherever possible, should be self-explanatory; hence the preference for the Page- Beberman symbols □ and △ to denote variables. (It seems reasonable to conjecture that if one left $3 + \square = 5$ written on the blackboard in a permissive second-grade classroom, some children would sooner or later write a "2" in the " □." Thus the problem of causing the children to get a certain experience without a prior use of English is by no means insuperable.) This kind of teaching strategy would seem to hold special promise for classes (as in East Africa) which are multi-lingual, and for classes of deaf-mute children. Even in such settings this kind of strategy is a relative novelty.

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[113] A conspicuous exception that does reflect such an approach in the sequence of pamphlets entitled *Experiences in Mathematical Discovery*, published by the National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W., Washington, D.C. 20036. for use with low-achieving students at approximately the grade nine level.

There is the further implication that "large" mathematical tasks should be broken into quite small ones, and these small tasks should be sequenced very carefully. (This, too, is often discussed in certain academic circles today, but it finds little practical expression either in our schools, or in programs of student teaching, teacher education, etc.)[1]

Curriculum 2: The new direction toward which the Project turned was based upon the idea that if some of the children in the lowest track seventh grade class could be helped to learn a significant portion of college-preparatory algebra and analytic geometry -- provided that a suitable pedagogical approach was employed -- then it might be worthwhile exploring how much untapped reserve ability was available in other student populations -- for example, in bright elementary school children, or in seventh-graders who had been diagnosed as "college capable."

Consequently, for the following two years a modified version of "Curriculum 1" was used with a special class of gifted third graders, working with Mrs. Pauline Stanley,[114] in a neighborhood that was ethnically described as "predominantly Italian," and a second section -- using quite similar content and pedagogical procedures -- was set up for a seventh grade class at the T. Aaron Levy School, in a predominantly professional neighborhood east of Syracuse University. Of the two classes, the gifted third graders probably displayed more mathematical insight and considerably greater motivation.

Both classes possessed far greater verbal ability than the Madison School children, and consequently the language of Curriculum 2 came to be more sophisticated than the language of "Curriculum 1." The unanticipated success with the gifted 3rd graders suggested that further explorations might profitably be directed toward bright elementary school children.

Unfortunately -- mainly by inadvertence -- in moving into this new situation and creating "Curriculum 2," the Project lost sight of individualized instruction and resorted almost exclusively to a teacher standing in front of the room, working with an entire class, although the actual atmosphere was kept quite informal and the room was usually the scene of lively, noisy, active student involvement in the task at hand.

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[114] These class trials with gifted 3rd graders were arranged by the Syracuse University Special Education Division, and by Dr. Gerald Cleveland of the Syracuse Public Schools.

"Curriculum ϵ." Chronologically, the third curriculum is the "sophisticated" curriculum discussed elsewhere in this report as "Curriculum ϵ." This curriculum resulted from the accident of the Project Director obtaining a leave of absence from Syracuse University and spending academic year 1959-1960 as a Visiting Lecturer at Yale University, and Assistant to the Director of the School Mathematics Study Group.

This led to running Madison Project trials in several public and private schools in the area of New Haven, Connecticut, some in the area of Hartford, Connecticut, and a particularly intensive program in Weston, Connecticut.

Weston, Connecticut, is an "ex-urb" of New York City, with a two-acre minimum plot size for home building, and with a prosperous and well-educated parent group. Many fathers in Weston are technically educated people (engineers, chemists, etc.), now employed in management positions in major corporations in New York City; another larger group of parents -- this time wives as well as husbands -- are involved with the media industry, or with creative work in the arts. There are editors, authors, painters, composers, television writers, producers, and directors, and film makers, along with designers, actors, illustrators, and others. The total population of Weston in 1959 was about 3,000 persons.

The program in Weston operated on the basis of a single one-hour long lesson once a week for each participating class, taught by a visiting specialist teacher. Comparison of Project results in different schools strongly suggests that if the specific abilities of the visiting specialist teacher are important to the success of such an operation, even greater importance should be attached to a "building coordinator" for the program, who is available five days per week, every week, to work with teachers, children, administrators, guidance personnel, and parents, and who concentrates all of her energies on one single school building. The program coordinator for the Weston Elementary School was Mrs. Beryl S. Cochran, and she had the highly effective cooperation of Principal Gilbert Brown, and Guidance Counsellor Herbert Barrett, an experienced clinical psychologist.

This team and setting combined to provide possibly the most effective educational environment the Project had found up until then -- or has found since -- and to this there was added the obviously superior verbal ability of Weston children, and their generally high motivation, as well as their tendency to feel at home in an abstract world. From this combination "Curriculum ϵ " was produced. Of all Project curricula, "Curriculum ϵ " is the most fully documented. For five years, every single Madison Project class taught in Weston was recorded on audio-tape, and these tapes have been preserved. In addition, a large number of lessons of "Curriculum ϵ " have been recorded on video-tape and 16 mm. film. "Curriculum ϵ" operated to a greater or lesser extent elsewhere at various times, including schools in: New Haven, Connecticut; North Haven, Connecticut; Scarsdale, New York; West Irondequoit, New York; Greece, New York; Lexington, Massachusetts; Clayton, Missouri; Webster Groves, Missouri; University City, Missouri; and Ladue, Missouri. However, the heights of sophisticated mathematics were scaled more effectively in Weston, Connecticut, than anywhere else at this time. (Portions of "Curriculum ϵ " had their second greatest success -- after Weston, and perhaps equal to or better than Weston -- in three ostensibly "culturally-deprived" areas: the Banneker District of St. Louis, the Admiral Richard E. Byrd School in Chicago; and the Model School District in Washington, D.C. Why these culturally-deprived areas should out-perform the average of a group of "culturally privileged" areas in dealing with sophisticated abstract mathematics is presently a matter for speculation -- by no means uninteresting, as a matter of fact.)

As mentioned earlier, in creating "Curriculum 2" the Project had abandoned the individualized instruction of "Curriculum 1" -- this change having been made more by accident than by design -- and came to place almost complete reliance on a teacher standing in the front of the room, working with a total class. The description given earlier, of "informal, lively, noisy, active student involvement in the task at hand" probably applies more to "Curriculum ϵ " than to any other Project program, although toward the end of the "Curriculum ϵ " stage, as a result of the Herbert Barrett study mentioned in Appendix A (and also following teacher recommendations), the Project introduced some use of physical materials, and returned to individualized instruction and independent work by individual students in those lessons where apparatus was involved. The most prominent apparatus used, in addition to simple measuring apparatus such as twelve inch rulers, meter sticks, protractors, etc., were

P.S.S.C. apparatus related to kinematics,[115] and Cuisenaire rods (or "centimeter blocks").

The main grade level focus of "Curriculum ϵ" is on grades four, five, and six, but it does extend as far as to range from the beginning of grade two to the end of grade nine.

By June, 1964, when the program in Weston ceased to be the main focus of most Project innovation, "Curriculum ϵ" had grown quite large, and was showing signs of breaking apart into several separate curricula, each intended to serve different purposes. These separate pieces gradually achieved separate identities, and came to differ in pedagogical methodology as well as in content.

At the time, and particularly if you focussed attention on Weston, Connecticut, it seemed that "Curriculum ϵ" was becoming more sophisticated and more abstract, moving from axiomatic algebra (a kind of modernized version of traditional ninth-grade algebra) taught in grade five, to matrix algebra and linear algebra in grade six, to a careful ϵ, N theory of limits of infinite sequences in grade eight.

In retrospect, it seems more accurate to say that several things were happening simultaneously. The use of Project materials in a rapidly increasing number of schools, and the consequent "viability testing" of units, was leading to a distinction between the "hothouse orchid" part of Curriculum ϵ, which functioned well only in the hands of a very few teachers, and only in a few educational settings (which, besides Weston, mysteriously included culturally-deprived urban schools in St. Louis, Chicago, and Washington, D.C., but did not include many other privileged schools), from a "hardy, durable" portion of Curriculum ϵ, that seemed to work nearly everywhere. Simple analytic geometry around grade three, four, or five worked almost everywhere. The arithmetic of signed numbers, at these same grade levels (three, four, and five) worked almost always and almost everywhere. Empirically-determined functions involving measurement uncertainty did not work everywhere, by any means, thanks to the unhappy popular prejudice that "every mathematics problem has exactly one right answer that is perfectly precise and absolutely correct." (Something like "tolerance for ambiguity," or even mature flexibility, is involved here.) Longer axiomatic proofs of algebraic theorems worked beautifully if approached in the right spirit, but as George Arbogast correctly observed, if taught in the wrong way this topic was as bad as Latin grammar when that was taught in the wrong way -- a lot of formal reading and writing with little about it that was creative or exciting or even meaningful. (This is too bad; as the tape-recorded lessons show, the 5th and 6th graders in Weston found the making of original axiomatic proofs of algebraic theorems to be one of the most exciting things they did in school, which means that they enjoyed it very much indeed, and they showed great ability both in conjecturing theorems and in proving them.) Obviously, finding matrix inverses and proving theorems about limits of infinite sequences by careful logic were not things to be attempted by any teacher and any group of students.

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[115] Cf., for example, the Madison Project film entitled Geoff's Experiment, or the film entitled One View of the New Mathematics Curricula.

Thus two tendencies were at work: one was to develop an increasingly sophisticated curriculum, the second was to separate this curriculum into a "stable" part and an "unreliable" part as a result of the program of viability testing. And, as we have seen, a third tendency had in fact entered the picture, with the increasing use of manipulatable physical apparatus and small-group or individualized instruction. From these divergences new curricula were born.[116]

Curriculum α. This curriculum was created from the stable portion of Curriculum ϵ, and forms the basis for the N.S.F.-sponsored In-Service Course I, which combines written material and films showing actual classroom lessons, in order to provide a program for teacher education. At the time that the "big city" workshops began in Chicago in 1964, Curriculum α formed the basis for the teacher education program.

Curricula β, γ, γ , and δ. The evolution of Curriculum β from

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[116] However, before this occurred, the two Discovery books were written, on the basis of experience acquired in Curriculum 1, Curriculum 2, and Curriculum ϵ.

Curriculum α, the creation of the various primary curricula γ, γ_1, etc., and the creation of the ninth-grade course, Curriculum δ, have been discussed at length elsewhere in this report, and also in various special reports that are available from the Project.

To give some idea of the contents and emphases of Curriculum β, we reproduce here an "inventory" used in the N.S.F.-sponsored "big city" workshops that presently follow the general lines of Curriculum β:

A. General Curriculum Inventory

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
1. counting, using discrete objects such as bottle caps, pebbles, etc.			at least as early as kindergarten or grade one	
1-A. using counting in relation to addition			at least as early as grade one	
1-B. using counting in relation to multiplication				
1-C. using counting in relation to division				
1-D. using counting in relation to subtraction				
2. rectangular arrays and multiplication			at least as early as grade two for most students	
3-A. the number line, for non-negative integers			at least as early as grade two	
3-B. the number line, for non-negative integers and simple fractions				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
3-C. the number line, for positive and negative integers				
4. informal preliminary experiences with fractions			at least as early as grade 2, and continuing to build on it for several subsequent years	
5-A. plotting points in Cartesian coordinates, in the first quadrant			at least as early as grade 2	
5-B. plotting points in Cartesian coordinates in all four quadrants				
6. use of □ and △ for variables			at least as early as grade 2	
7. truth sets for open sentences				
8. explicit discussion of replacement sets for variables				
9. linear graphs for addition facts			at least as early as grade 2	

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
10. measurement by ordering				
A. length				
B. area				
C. volume				
D. angles (regarded as rotations)				
E. angles (defined by two rays)			in secondary school, but not in elementary school	
F. time				
G. weight				
H. density				
I. perimeters, circumferences, and similar "girth" types of measures (e.g. measuring children's waist sizes)				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the *Madison Project* recommends introducing this topic...	Comments
11. measurement by inequalities, with non-standard units				
A. length				
B. area				
C. volume				
D. angles (rotations)				
E. angles (rays)			in secondary school, but not in elementary school	
F. time				
12. measurement by inequalities, with commonly-shared (but un-named)units				
A. length				
B. area				
C. volume				
D. angles (rotations)				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
E. angles (rays)			in secondary school, but not in elementary school	
F. time				
G. weight				
H. perimeters, circumferences, etc.				
13. measurement by inequalities, using standard units				
A. length				
B. area				
C. volume				
D. angles (rotations)				
E. angles (rays)			in secondary school, but not in elementary school	
F. time				
G. weight				
H. density				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
14-A. grouping, and place-value numerals, base three				
14-B. grouping, and place-value numerals, bases 4, 5, and 6				
14-C. grouping, and place-value numerals, base 10				
15-A. "guess my rule" (i.e., "guessing functions") with rules like $\square + 1 = \triangle$				
15-B. "guess my rule," with more complicated rules				
16. linear graphs, including the concept of slope (and possibly also y-intercept)				
17-A. classification, using attribute blocks				
17-B. classification, using letters of the alphabet				
17-C. classification, using everyday objects				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
18. explicit consideration and use of "P.N."				
19. explicit consideration and use of "U.V."				
20. operations with signed numbers			not later than grade 5	
21. classification of open sentences as identities or not				
22. making up identities				
23. accumulating a list of identities				
24. derivations of identities				
25. "shortening lists" of identities				
26. identifying axioms and theorems				
27. explicit consideration of the logical rules used in making derivations				
28. "functions" vs. "formulas"				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
29. empirically-obtained functions represented by graphs				
30. non-linear graphs, especially conic sections				
31-A. a primary grades "math lab" within the regular classroom				
31-B. primary grades math lab as a separate room, with regular classroom teachers bringing their children				
31-C. primary grades math lab as a separate room, with a special "math lab" teacher				
32-A. intermediate grades math lab within the regular classroom				
32-B. intermediate grades math lab as a separate room, with regular classroom teachers bringing their children				
32-C. intermediate grades math lab as a separate room, with a special "math lab" teacher				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
33-A. junior high math lab, as part of the regular classroom				
33-B. junior high math lab, as a separate room, with regular classroom or math teachers bringing their children				
33-C. junior high math lab, as a separate room, with a special "math lab" teacher				
34. finite difference methods for classifying functions				
35-A. students determine their own truth values and make up their own truth tables				
35-B. standardization of truth tables from 35-A				
35-C. inference schemes				
35-D. truth tables as mappings				
36-A. adding and multiplying 2 x 2 matrices			not recommended except for capable students and specially-educated teachers	
36-B. finding the unit matrix				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth, the Madison Project recommends introducing this topic...	Comments
36-C. complex numbers via matrices			not recommended except for capable students and specially-educated teachers	
37-A. statistics: frequencies				
37-B. statistics: distributions				
37-C. statistics: measures of central tendency				
37-D. statistics: measures of dispersion				
37-E. statistics: sampling				
38. the concept of area and heuristic arguments for finding area				
39. formulas for area				
40. indirect measurement via scale drawings				
41. trees				
42. mappings				

Topic	Would you include it or not?	At what grade levels would you teach it or subsequently use it?	For what it's worth the Madison Project recommends introducing this topic...	Comments
43. applications of trees, mappings, etc., in Wylie's "101 Problems in Thought and Logic" (Dover, paperback)				
Your suggestions:				

B. Specific Teaching Devices

Device	Would you include it?	At what grade levels?	For what it's worth, the Madison Project recommends ...	Comments
1. Pebbles-in-the-Bag model to introduce signed numbers			introduction not later than grade 5, and preferably earlier	
2. Identification of three roughly-separated number systems: i) the "counting" or "how many" system ii) the "sharing" or "measuring" system iii) the "reference-point" system				
3. Postman stories to introduce binary operations on signed numbers				
4. Quadratic equations for drill and practice in arithmetic, for practice using variables, and for discovery opportunities				
5-A. Dienes' MAB blocks, base 3				

Device	Would you include it?	At what grade levels?	For what it's worth, the Madison Project recommends ...	Comments
5-B. Dienes' MAB blocks, bases 4, 5, and 6				
5-C. Dienes' MAB blocks, base 10				
6-A. Cuisenaire rods for fractions				
6-B. Cuisenaire rods for other purposes				
7. UICSM "shrinkers" and "stretchers" for fractions				
8-A. Plastic dial "adding machines"				
8-B. 10-key paper-read-out adding machines (electric)				
8-C. full keyboard desk calculator, manual				
8-D. full keyboard desk calculator, electric				
8-E. double keyboard desk calculator (traditional "statistical" model)				
9-A geoboards for concept of area				

Device	Would you include it?	At what grade levels?	For what it's worth, the Madison Project recommends ...	Comments
9-B. length; area: the "pretend" game vs. the "reality" game				
9-C. ability to copy patterns from geoboard to dot paper				
9-D. ordering the area of the states (Missouri, New York, Alaska, etc.), by using graph paper				
10. measurement error: central tendency and dispersion				
11-A. Tower Puzzle, for example of an exponential function				
11-B. "Shuttle puzzle," as example of a quadratic function				
11-C. Weights and springs				
12. "clues" or "hidden numbers"				
13. density as slope of graph				
14. velocity and acceleration, with PSSC approach				

Device	Would you include it?	At what grade levels?	For what it's worth, the Madison Project recommends ...	Comments
15. Mirror cards				

Your suggestions (note that we have not tried to spell out contents of "math labs" in detail, except for a few major items)

Appendix H

What Should a School Be Like?

There is a growing recognition that most of today's schools do not provide environments appropriate to effective learning. An increasing number of conferences have concerned themselves with this topic, and a few schools organized along quite different lines are in actual operation in a few places in the United States, and much more commonly in England.

The forthcoming report of the 1967 Summer Conference at Pine Manor, organized by E.D.C.'s Cambridge Conference on School Mathematics to consider the interrelation of mathematics and science, K-12 (and supported by the National Science Foundation), will probably present several interesting models for different types of schools, including an especially provocative suggestion by Jack Easley. The articles by Joseph Featherstone, and the article by Dean Schaefer, cited elsewhere in this report, also discuss the question of what a "school" should look like and be like.

Work in curriculum and instruction is inevitably related to assumptions about the nature of schools, and to the extent that any specific curriculum and methodology is implemented, it necessarily creates pressures that tend to shape schools in a certain way.

The curricula developed by the Madison Project have been, and can be, used in a wide variety of instructional settings; however, the Project is coming increasingly to give serious thought to the new kinds of schools which are emerging, in England and in the United States. We reproduce here, with permission, a short excerpt dealing with the nature of improved schools, that was written by Peter Shoresman of the University of Illinois at Urbana, and appeared as one of the working papers at the Pine Manor Conference:

THE INSTRUCTIONAL SETTING[1]

"It should be a rather obvious axiom of education that the instructional setting should provide conditions which facilitate learning -- not hinder it. If a choice must be made, this setting should be modeled around sound principles of learning rather than around sound principles of discipline (although accomplishment of the former often leads to easy accomplishment of the latter). Furthermore, the instructional setting should be geared to the student first and the teacher second. (It is not necessarily true that easy ways of teaching make available easy ways of learning.)

"The outstanding characteristic of the instructional setting should be its ability to cope flexibly with the learning and social needs of children. This flexibility has spatial, human, and temporal dimensions. First, children should be free to move from one learning facility to another -- for example, from classroom to laboratory to library. Provisions should also be made for placement of children within organizational frameworks which are most conducive to the particular task they are undertaking or to the particular problems they are encountering. For some purposes, children need to work in small groups; for others, in large groups; and for still others, individually.

"Secondly, children should have the opportunity to come in contact with teachers with expertise in a given area of the curriculum. Somewhere in the school, there should be a teacher who knows a great deal about science and enjoys teaching it; another teacher who knows a great deal about mathematics and enjoys teaching it; and so on.

"Thirdly, children should be allowed much more temporal flexibility than is currently the practice. Time scheduling during the school day and week should be flexible enough to permit them to spend a greater amount of time -- in blocks of sufficient length -- to pursue a topic in which they have become intensely interested. From a broader point of view, schools should be so organized that children are not required to follow the lock step of grade level placement of subject matter. Movement from one concept to another, and from one topic area to another, should occur as readiness and ability indicate. Here and there throughout the country, a number of schools are experimenting with organizational structures which provide models for the establishment of such flexibility. Variations of a team teaching/nongraded structure appear to be among the most promising of present models."[117]

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[117] This excerpt is quoted from: Peter Shoresman, "A Type of Essay -- An Immodest Proposal: Back to Natural Philosophy," a working paper of the 1967 Pine Manor Conference of Education Development Center, Inc., sponsored by the National Science Foundation.